非可換量子渦
Non-Abelian Vortex

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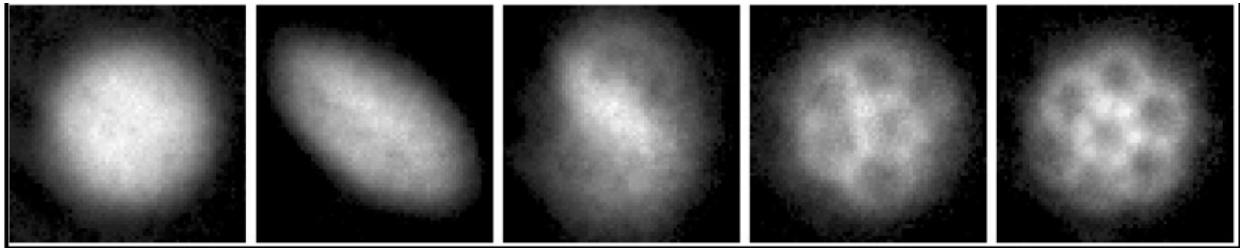
Mar. 28, 2009, 日本物理学会第64回年次大会

Conclusion

1. **Non-Abelian Vortices are realized in the cyclic phase of spin-2 Bose-Einstein condensates**
2. **Non-Abelian character becomes remarkable in collision dynamics of two vortices**
   I. **We numerically show.**
   II. **We algebraically confirm.**

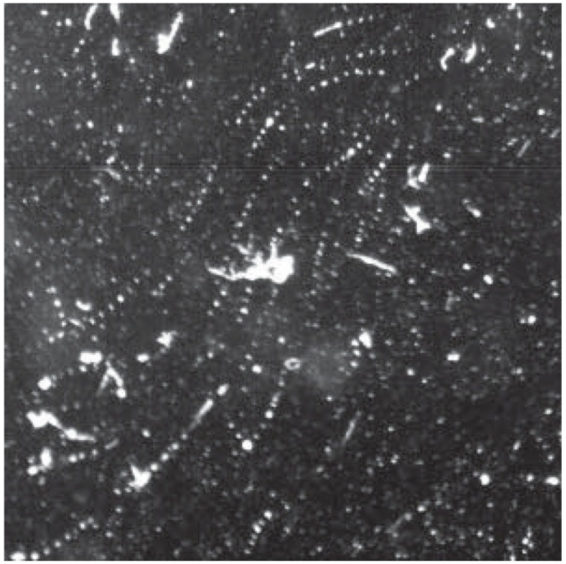
Vortex in Bose-Einstein Condensates



vortex in ^{87}Rb BEC

K. W. Madison et al.
PRL **86**, 4443 (2001)

vortex in ^{4}He



G. P. Bewley et al.
Nature **441**, 588 (2006)

Vortices appears as line defects when symmetry breaking happens

- Vortices are Abelian for single-component BEC
- **We here consider vortices called "Non-Abelian"**

Quantized Vortex and Topological Charge

Topological charge of a vortex can be considered how order parameter changes around the vortex core

Single componen ...]

Non-Abelian vortices are realized in the cyclic phase of spin-2 BEC

... be expressed

... integer n (: $\pi_1(G/H) = Z$)

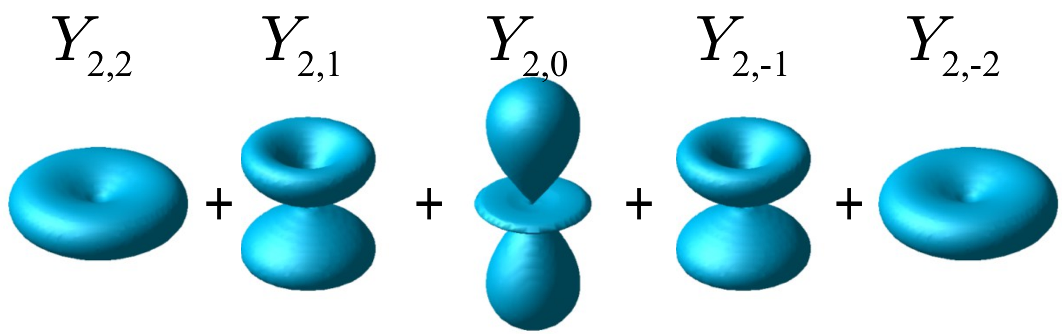
When topological charge can be expressed by non-commutative algebra (: first homotopy group π_1 is non-Abelian), we define vortices as "**non-Abelian vortices**"

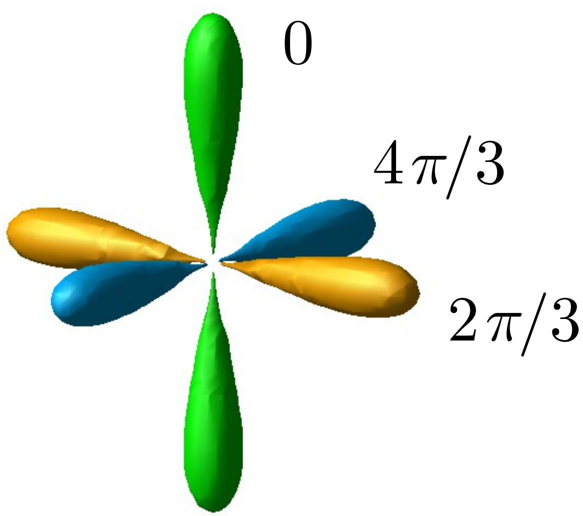
Spin-2 Bose-Einstein Condensate

cyclic phase

$$\sum_{m=-2}^{2} \Psi_m Y_{2,m}$$

$$\underbrace{e^{i\phi}}_{\text{gauge}} \underbrace{e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{F}}\alpha}}_{\text{spin}} \begin{pmatrix} i/2 \\ 0 \\ 1/\sqrt{2} \\ 0 \\ i/2 \end{pmatrix}$$



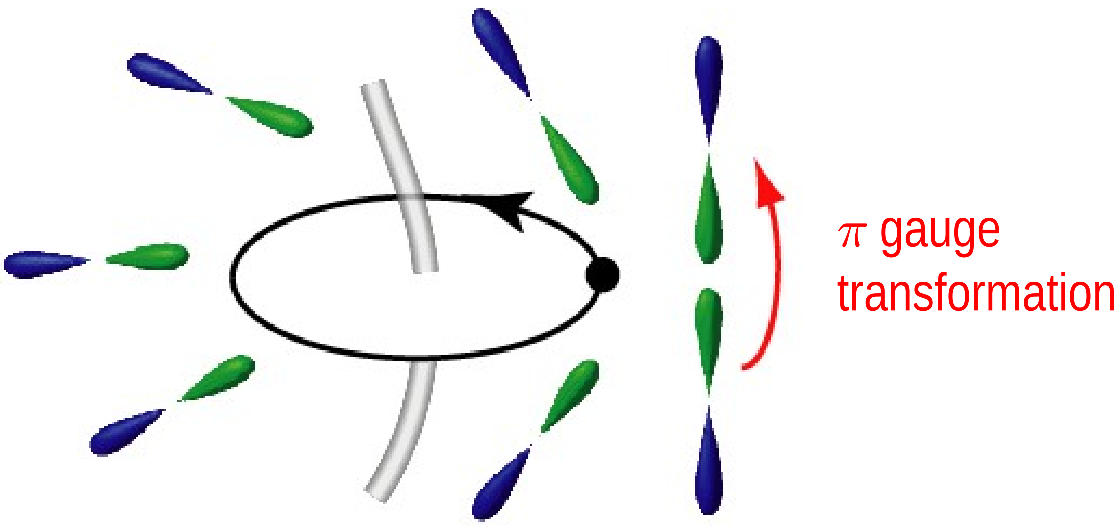


headless triad : axes can interchange each other by $2\pi/3$ gauge transformation (different from triad of ^{3}He-A)

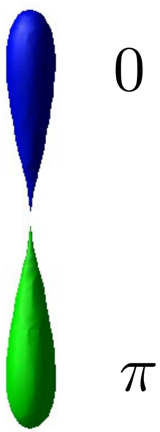
Vortices in Spinor BEC

$S = 1$ Polar phase

$$e^{i\phi}e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{F}}\alpha}\begin{pmatrix}0\\1\\0\end{pmatrix}$$



headless vector

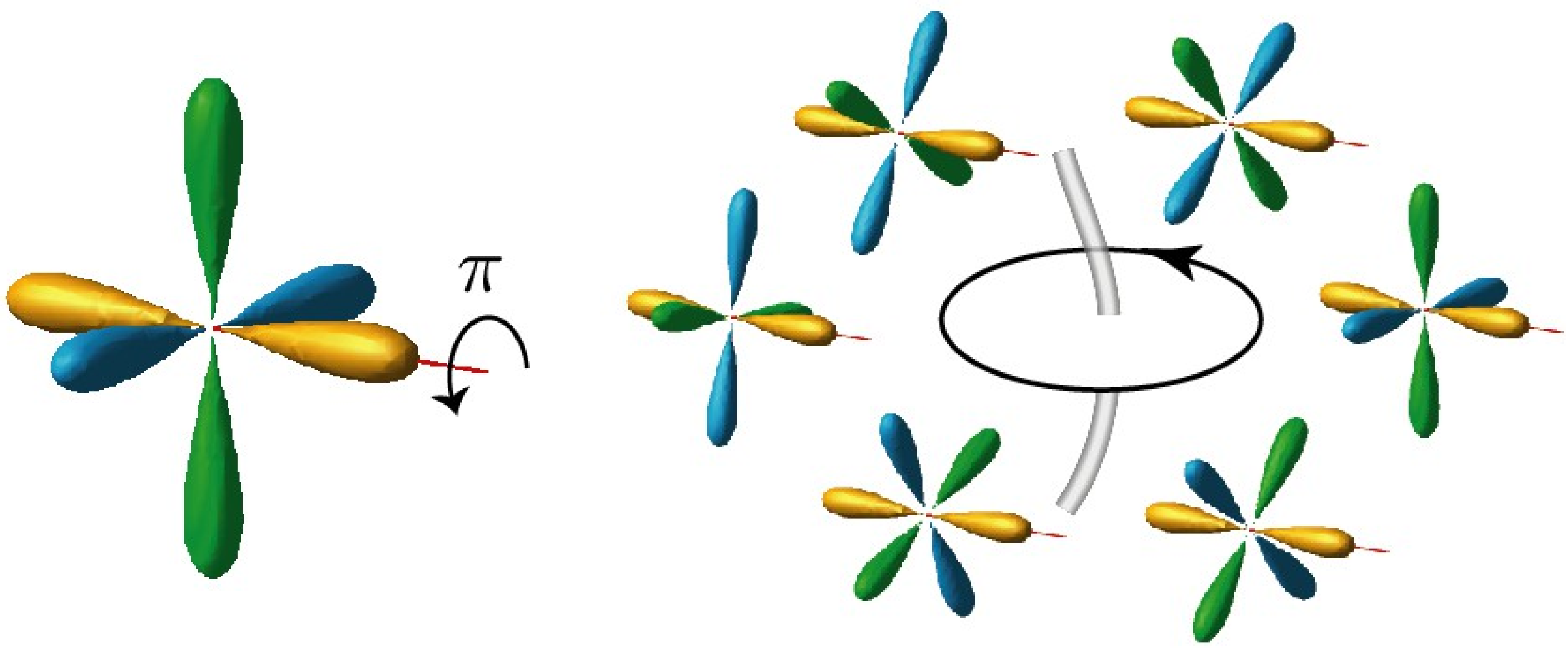


Half quantized vortex : spin & gauge rotate by π around vortex core

Topological charge can be expressed by half integer (Abelian vortex)

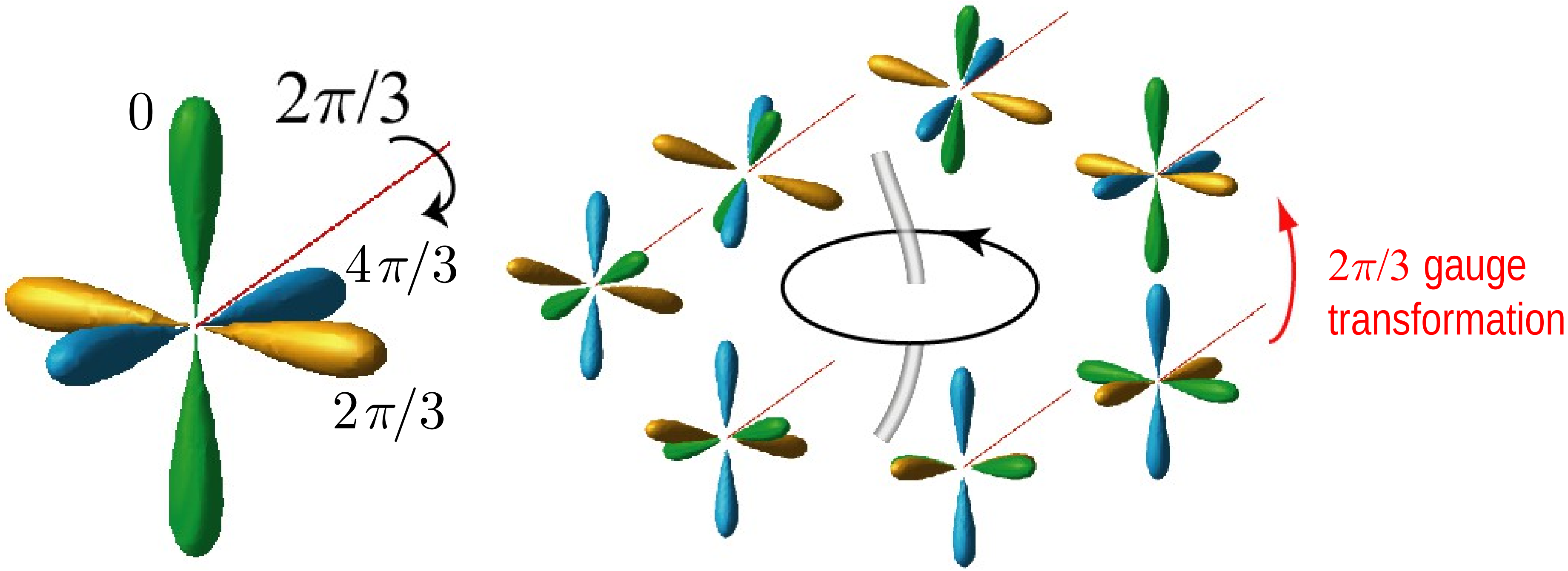
Vortices in Spin-2 BEC

1/2-spin vortex : triad rotate by π around three axis $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$



Vortices in Spin-2 BEC

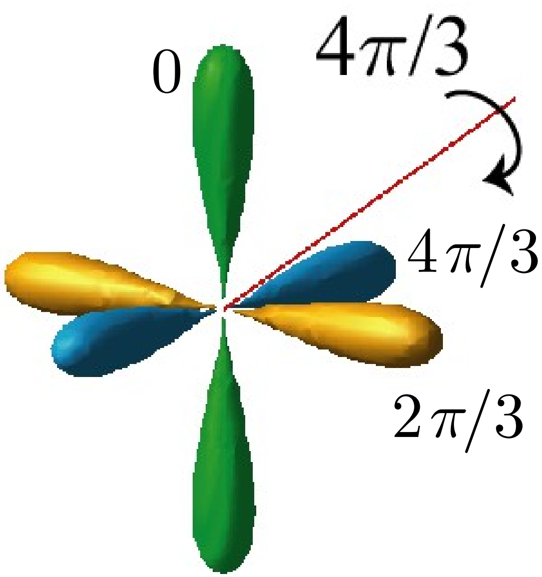
1/3 vortex : triad rotate by $2\pi/3$ around four axis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, $\boldsymbol{e}_4$
and $2\pi/3$ gauge transformation

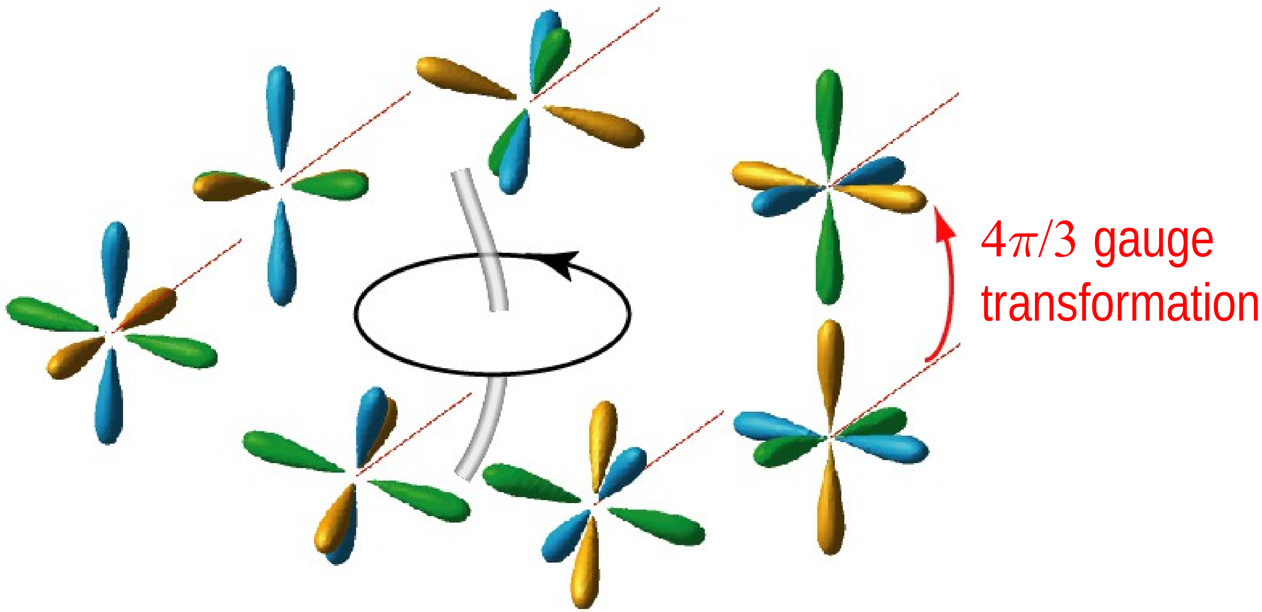


$$\boldsymbol{e}_1 = (1,1,1)/\sqrt{3}, \quad \boldsymbol{e}_2 = (1,-1,-1)/\sqrt{3}$$
$$\boldsymbol{e}_3 = (-1,1,-1)/\sqrt{3}, \quad \boldsymbol{e}_4 = (-1,-1,1)/\sqrt{3}$$

Vortices in Spin-2 BEC

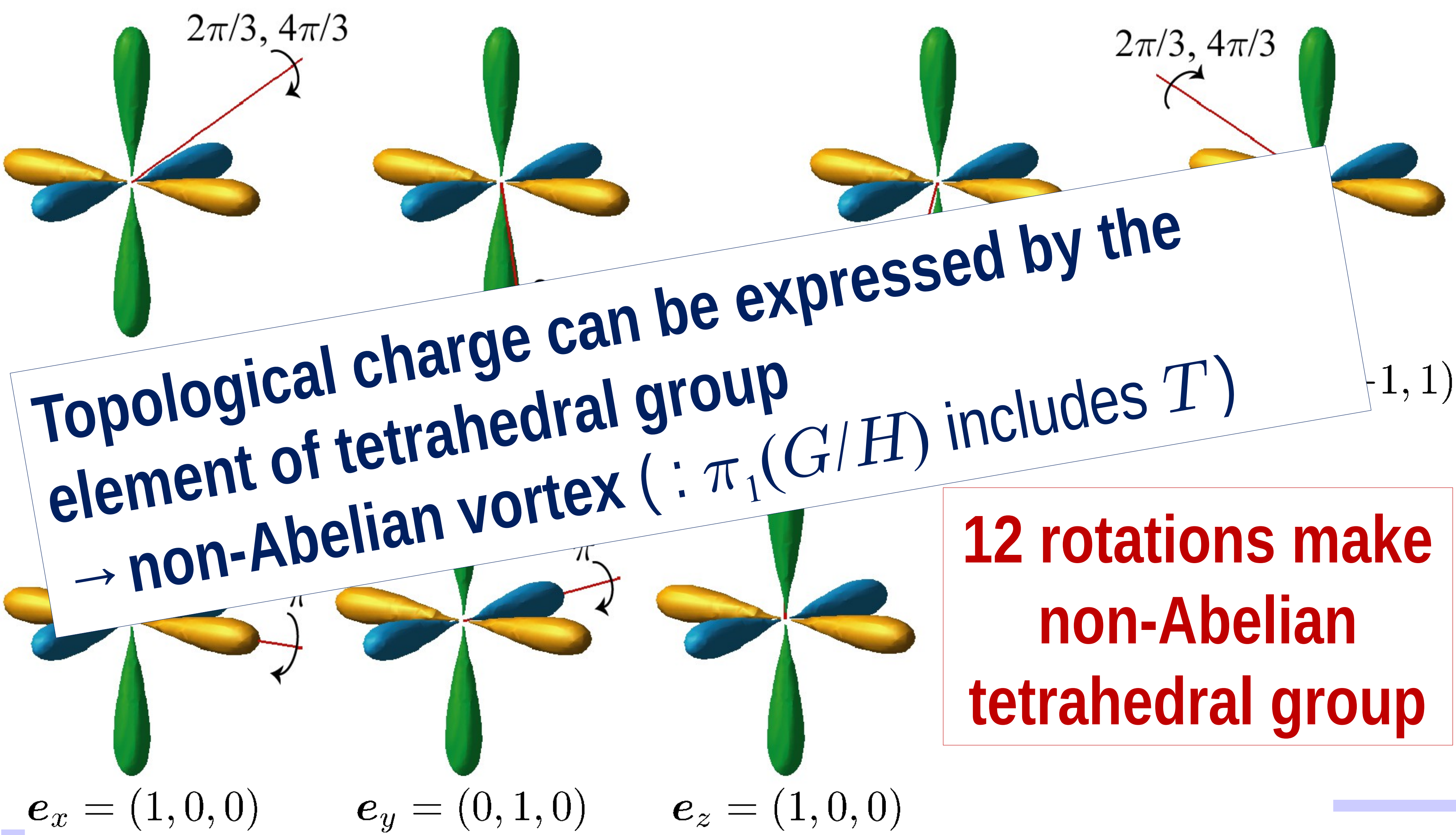
2/3 vortex : triad rotate by $4\pi/3$ around four axis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, $\boldsymbol{e}_4$ and $4\pi/3$ gauge transformation





$$\boldsymbol{e}_1 = (1,1,1)/\sqrt{3}, \quad \boldsymbol{e}_2 = (1,-1,-1)/\sqrt{3}$$
$$\boldsymbol{e}_3 = (-1,1,-1)/\sqrt{3}, \quad \boldsymbol{e}_4 = (-1,-1,1)/\sqrt{3}$$

Topological Charge of Vortices is Non-Abelian



Topological charge can be expressed by the element of tetrahedral group
→ non-Abelian vortex (: $\pi_1(G/H)$ includes T)

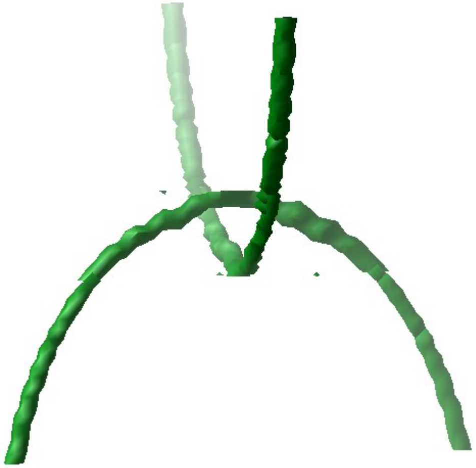
12 rotations make non-Abelian tetrahedral group

Collision Dynamics of Vortices

"**Non-Abelian**" character becomes remarkable when two vortices collide with each other

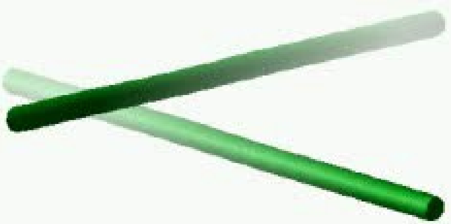
→Numerical simulation of the Gross-Pitaevskii equation

Initial state：two straight vortices in oblique angle, linked vortices



Collision Dynamics of Vortices

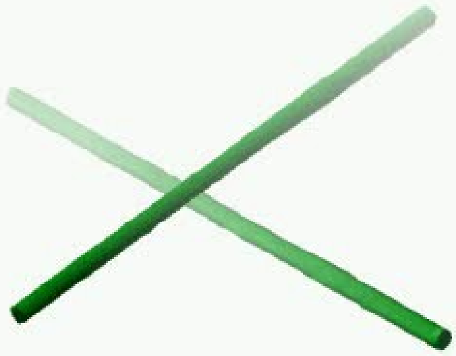
Commutative topological charge



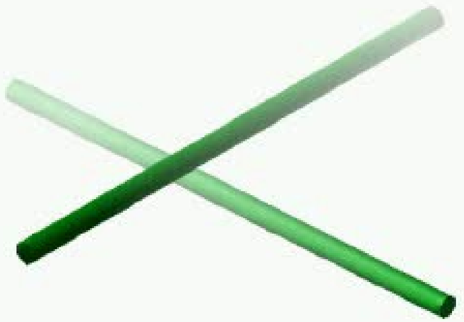
reconnection

passing through

Non-commutative topological charge



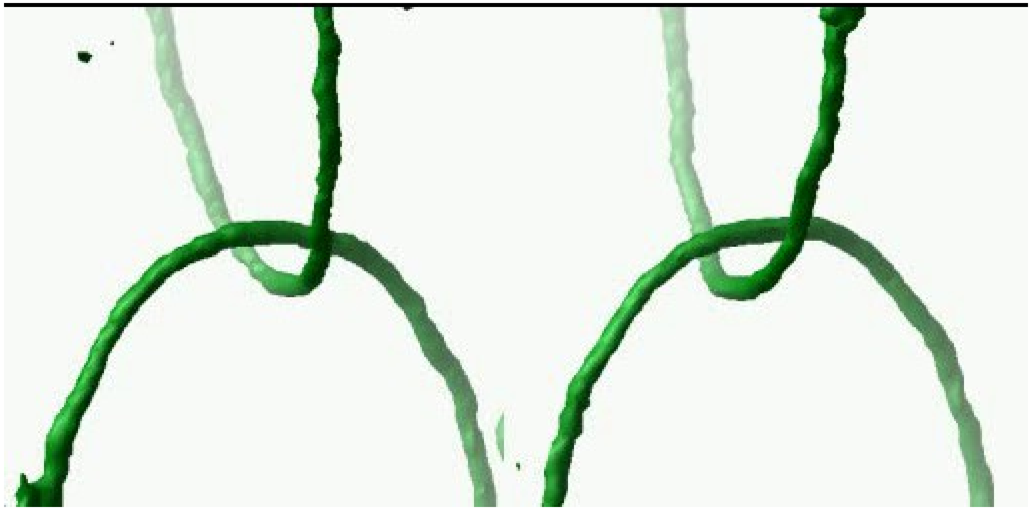
polar rung



ferromagnetic rung

Collision Dynamics of Linked Vortices

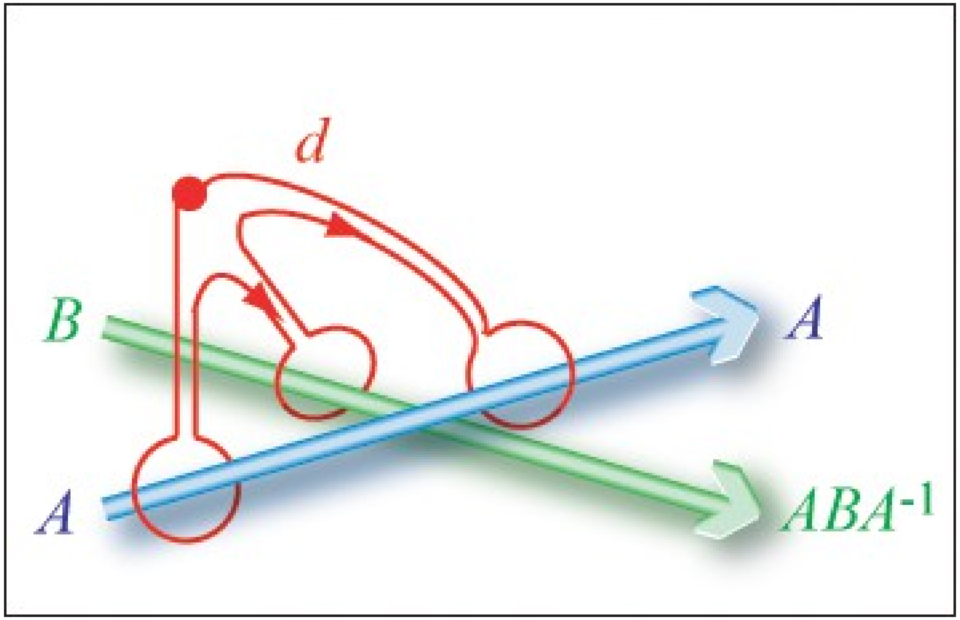
Commutative | Non-commutative



untangle | not untangle

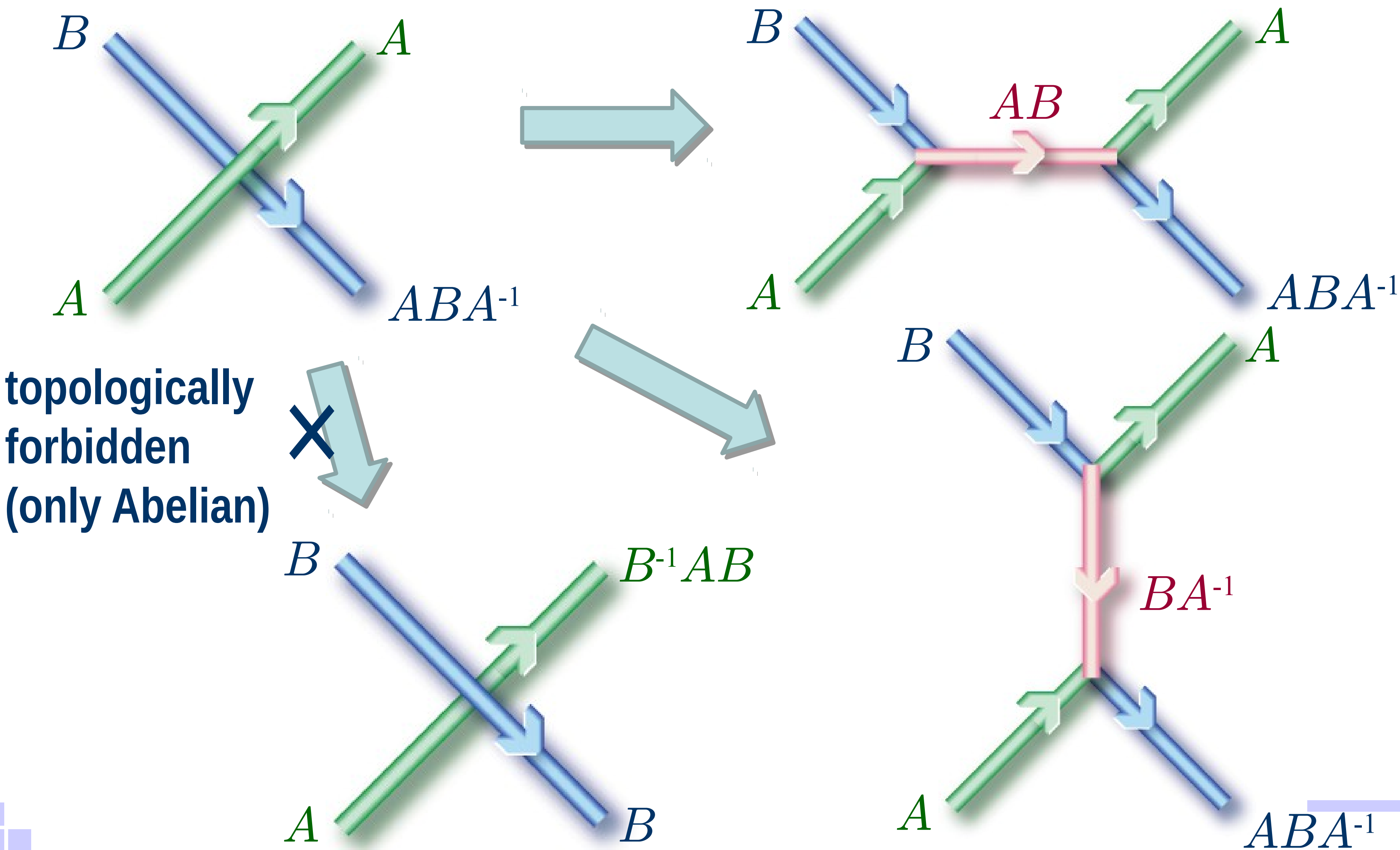
Algebraic Approach

Consider 4 closed paths encircling two vortices

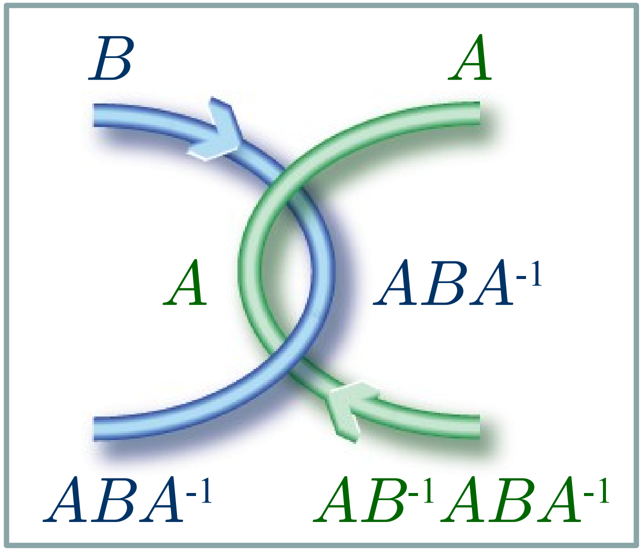


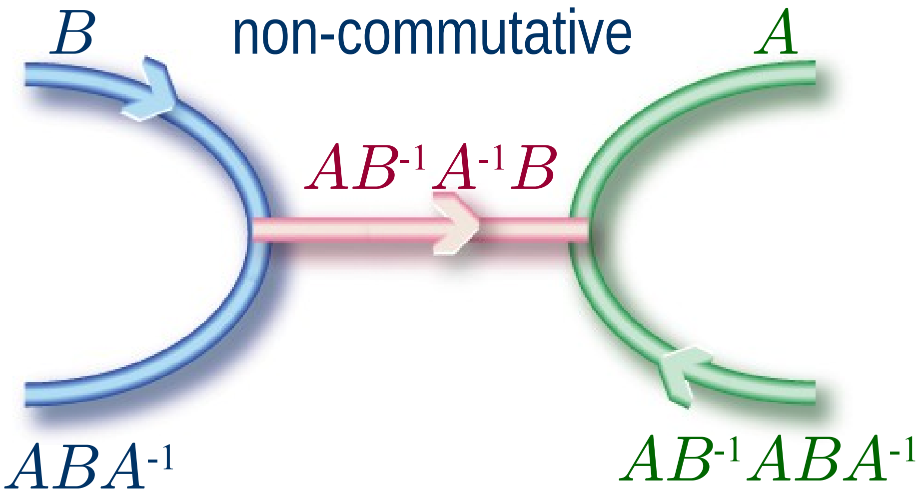
Path d defines vortex B as ABA^{-1} (same conjugacy class)

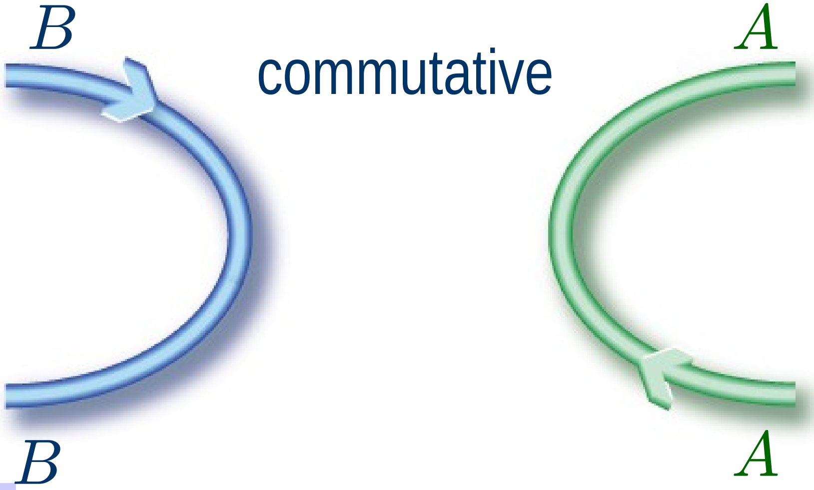
Collision of Vortices



Linked Vortices





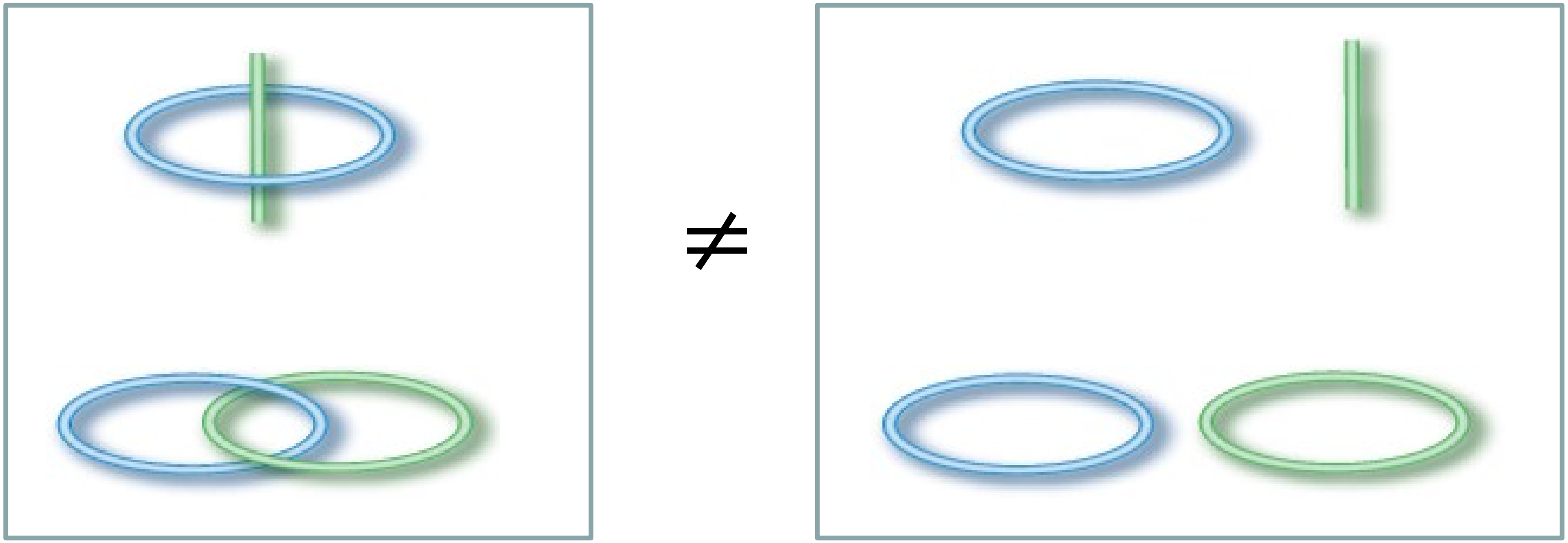


Linked vortices cannot untangle

Summary

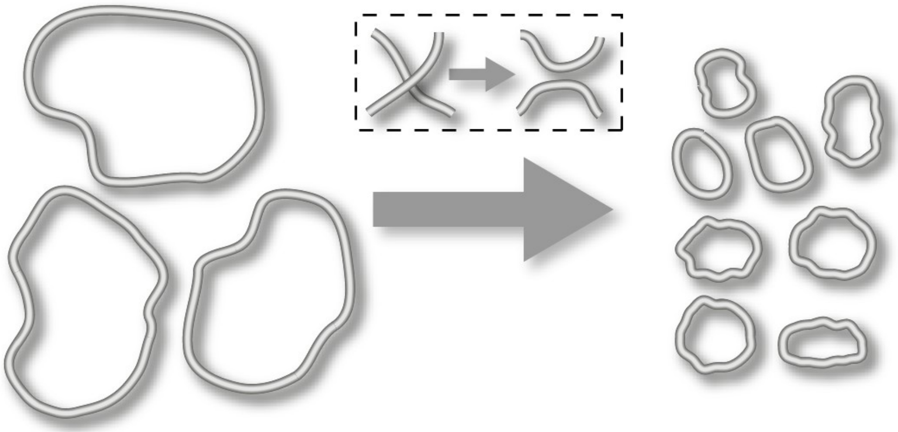
1. Vortices with non-commutative topological charge are defined as non-Abelian vortices.
2. Non-Abelian vortices can be realized in the cyclic phase of spin-2 BEC
3. Collision of two non-Abelian vortices create a new vortex between them as a rung (networking structure).

Future: Topological Charge of Linked Vortices



Do linked vortices themselves have a different topological charge from them as each vortices?

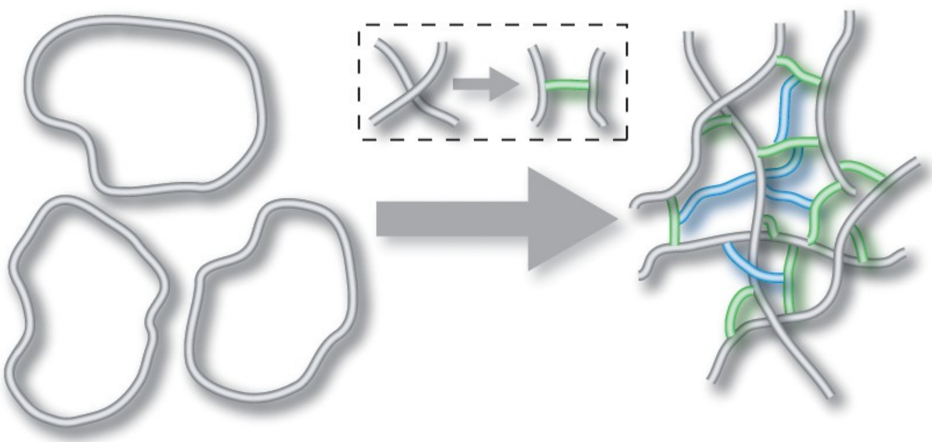
Future: Network Structure in Quantum Turbulence



Turbulence with Abelian vortices

↓

•Cascade of vortices



Turbulence with non-Abelian vortices

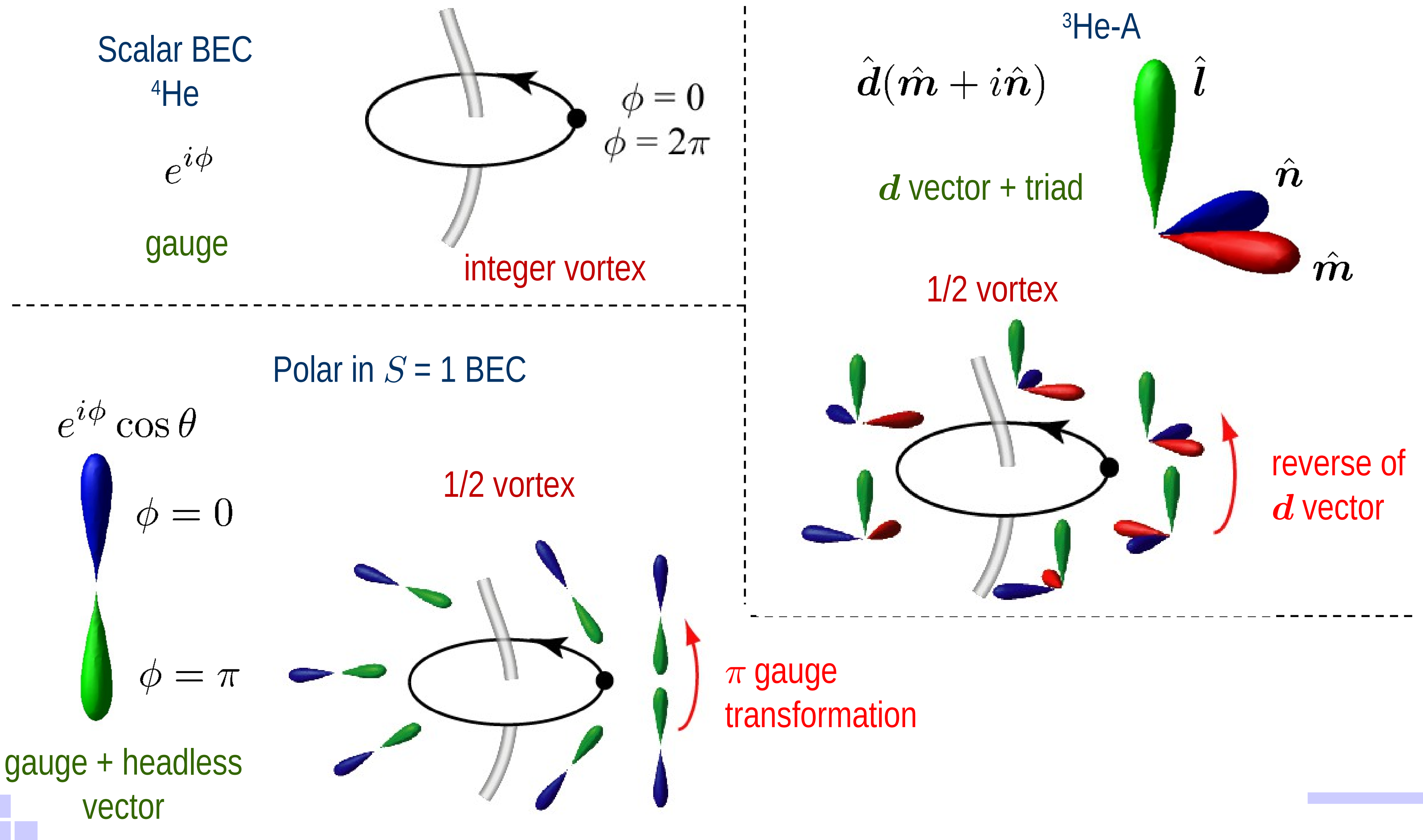
↓

•Large-scale networking structures among vortices with rungs

•Non-cascading turbulence

New turbulence!

Quantized Vortices in Multi-component BEC



Quantized Vortices in Multi-component BEC

Abelian vortices

scalar BEC
integer vortex

example :
$$\Psi \propto \exp[i\theta]$$

Polar phase in spin-1 spinor BEC
1/2 vortex

$$\Psi \propto \frac{1}{\sqrt{2}} \begin{pmatrix} \exp[i\theta] \\ 0 \\ 1 \end{pmatrix}$$

non-Abelian vortices

Cyclic phase in spin-2 spinor BEC
1/2 spin vortex or 1/3 vortex

$$\Psi \propto \frac{1}{2} \begin{pmatrix} i\exp[i\theta] \\ 0 \\ \sqrt{2} \\ 0 \\ i\exp[-i\theta] \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} \exp[i\theta] \\ 0 \\ 0 \\ \sqrt{2} \\ 0 \end{pmatrix}$$

We investigate detailed structure of vortices

Mean Field Approximation for BEC at $T = 0$

Spin-2

$$H \simeq \int d\boldsymbol{x} \left[-\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + \frac{c_0}{2} n_{\mathrm{tot}}^2 + \frac{c_1}{2} \boldsymbol{F}^2 + \frac{c_2}{2} |A_{00}|^2 \right]$$

$$c_0 = \frac{4g_2 + 3g_4}{7}, \quad c_1 = \frac{g_4 - g_2}{7}, \quad c_2 = \frac{7g_0 - 10g_2 + 3g_4}{7}$$

$$n_{\mathrm{tot}}(\boldsymbol{x}) = \Psi_m^*(\boldsymbol{x}) \Psi_m(\boldsymbol{x}), \quad \boldsymbol{F}(\boldsymbol{x}) = \Psi_m^*(\boldsymbol{x}) \hat{\boldsymbol{F}}_{mm'}(\boldsymbol{x}) \Psi_{m'}(\boldsymbol{x})$$

$$A_{00}(\boldsymbol{x}) = \frac{1}{\sqrt{5}} [2\Psi_2(\boldsymbol{x})\Psi_{-2}(\boldsymbol{x}) - 2\Psi_1(\boldsymbol{x})\Psi_{-1}(\boldsymbol{x}) + \Psi_0(\boldsymbol{x})^2]$$

n_{tot} : total density

$\boldsymbol{F}$: magnetization

A_{00} : singlet pair amplitude

Spin-2 BEC

$$H \simeq \int d\boldsymbol{x} \left[-\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + \frac{c_0}{2} n_{\rm tot}^2 + \frac{c_1}{2} \boldsymbol{F}^2 + \frac{c_2}{2} |A_{00}|^2 \right]$$

1. $c_1 < 0 \rightarrow$ ferromagnetic phase : $\boldsymbol{F} \neq 0$
2. $c_1 > 0,\ c_2 < 0 \rightarrow$ polar phase : $\boldsymbol{F} = 0,\ A_{00} \neq 0$
3. $c_1 > 0,\ c_2 > 0 \rightarrow$ cyclic phase : $\boldsymbol{F} = A_{00} = 0$

ferromagnetic

$$e^{i\phi} e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{F}}\alpha} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

polar

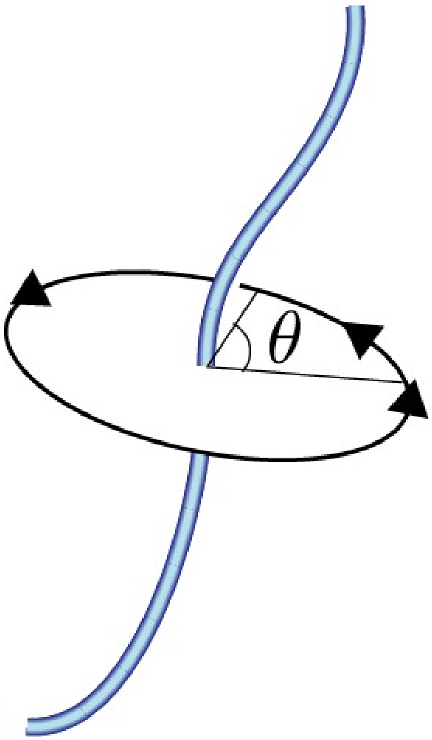
$$e^{i\phi} e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{F}}\alpha} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

cyclic

$$e^{i\phi} e^{-i\boldsymbol{e}\cdot\hat{\boldsymbol{F}}\alpha} \begin{pmatrix} i/2 \\ 0 \\ 1/\sqrt{2} \\ 0 \\ i/2 \end{pmatrix}$$

Vortices in Spin-2 BEC

Wave function of each vortex



gauge vortex

$$\hat{S}e^{i\theta}\begin{pmatrix} i/2 \\ 0 \\ 1/\sqrt{2} \\ 0 \\ i/2 \end{pmatrix}$$

integer-spin vortex

$$\frac{\hat{S}}{2}\begin{pmatrix} i\exp[2i\theta] \\ 0 \\ \sqrt{2} \\ 0 \\ i\exp[-2i\theta] \end{pmatrix}$$

1/2-spin vortex

$$\frac{\hat{S}}{2}\begin{pmatrix} i\exp[i\theta] \\ 0 \\ \sqrt{2} \\ 0 \\ i\exp[-i\theta] \end{pmatrix}$$

1/3 vortex

$$\frac{\hat{S}}{\sqrt{3}}\begin{pmatrix} \exp[i\theta] \\ 0 \\ 0 \\ \sqrt{2} \\ 0 \end{pmatrix}$$

2/3 vortex

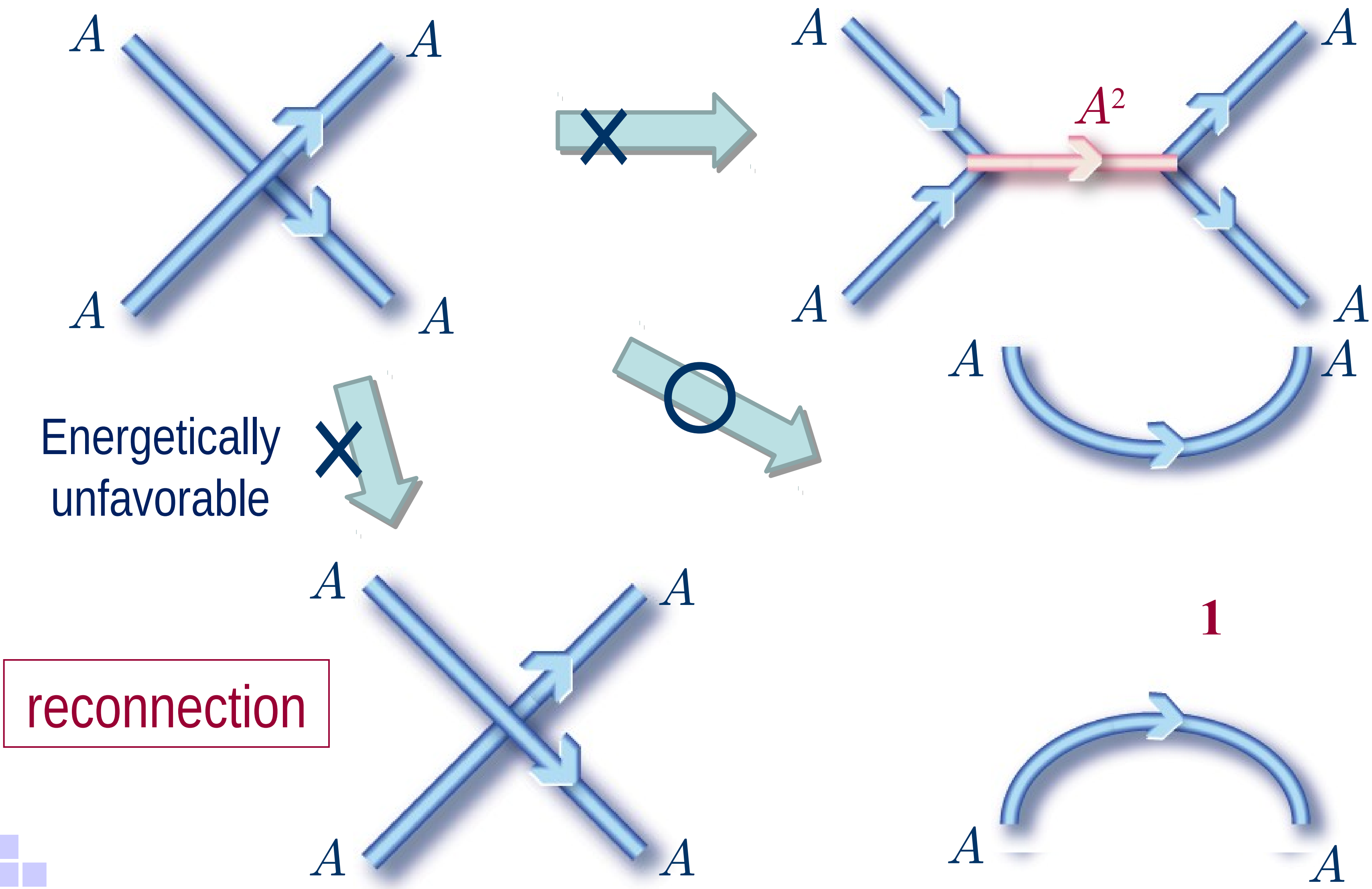
$$\frac{\hat{S}}{\sqrt{3}}\begin{pmatrix} \exp[2i\theta] \\ 0 \\ 0 \\ \sqrt{2} \\ 0 \end{pmatrix}$$

$\hat{S}$: arbitrary gauge transformation and spin rotation

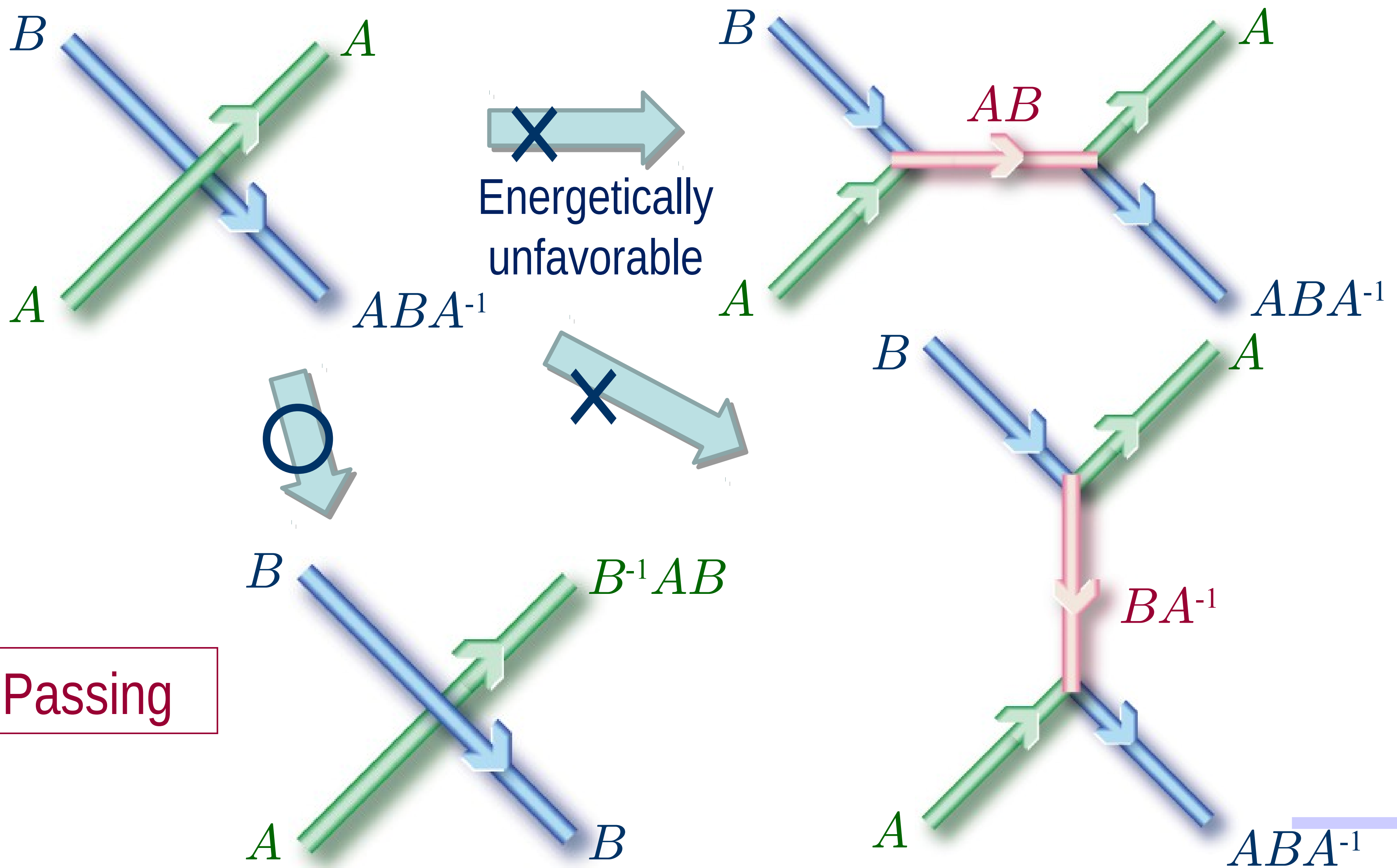
Vortices in Spin-2 BEC

vortices	gauge rotation	spin rotation	core structure
gauge	1	0	density core
integer spin	0	1	polar core
1/2 spin	0	1/2	polar core
1/3	1/3	1/3	ferromagnetic core
2/3	2/3	2/3	ferromagnetic core

Collision of Same Vortices

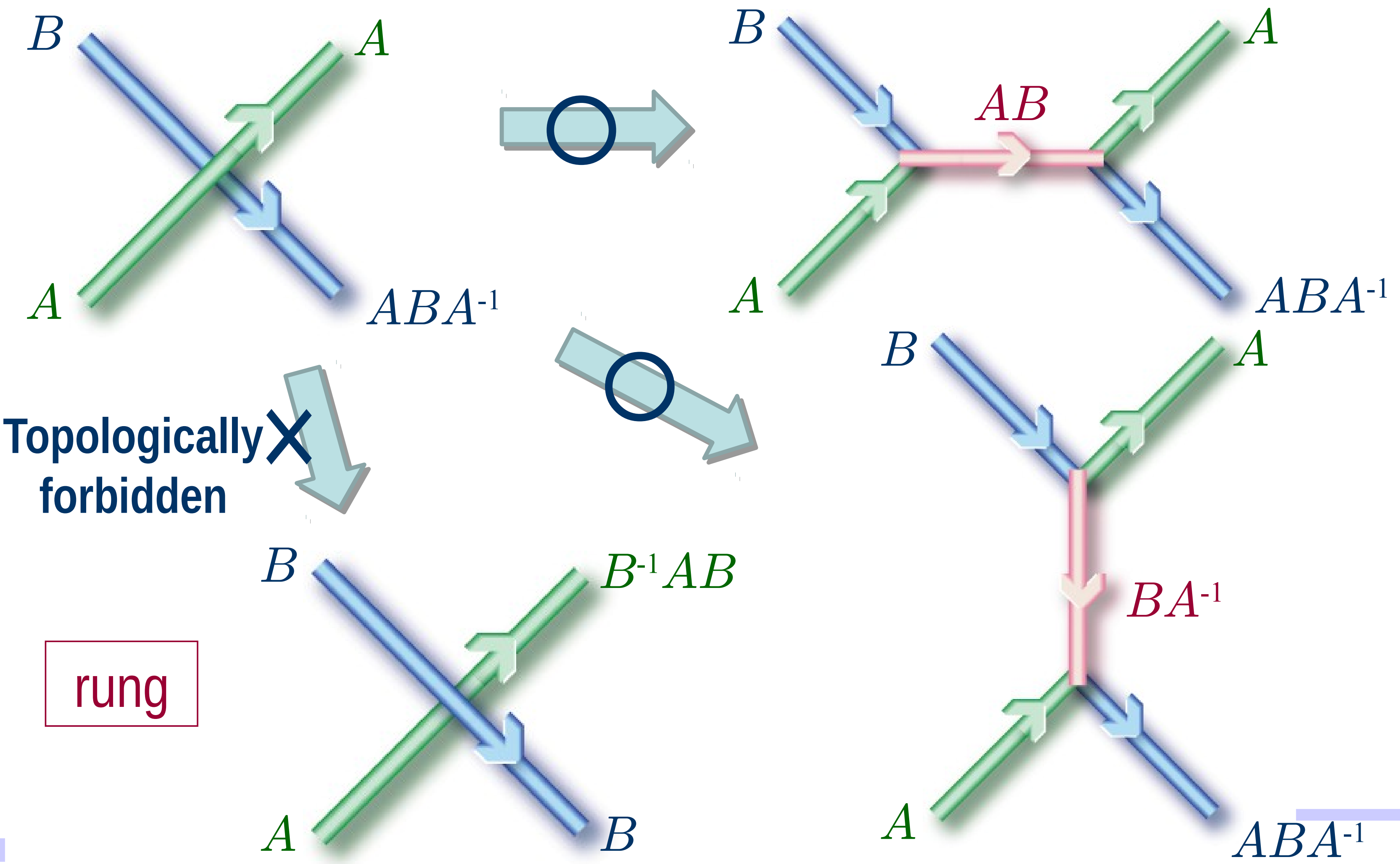


Collision of Different Commutative Vortices



Passing

Collision of Different Non-commutative Vortices



Homotopy Group of the Cyclic Phase of Spin-2 BEC

$$\text{Order parameter manifold}: \frac{G}{H} \sim \frac{U(1)_G \times SO(3)_S}{T_{G+S}}$$

$$\text{First homotopy group}: \pi_1\left(\frac{G}{H}\right) \sim \pi_1\left(\frac{U(1)_G \times SO(3)_S}{T_{G+S}}\right)$$

$$\sim \pi_1\left(\frac{U(1)_G \times SU(2)_S}{T^*_{G+S}}\right)$$

$$\frac{T^*}{Z_2} \sim T$$