Topology and topological defects appeari ng in Bose-Einstein condensate

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Model Equations in Bose-Einstein Condensation and Relate topics (Dec. 7, 2010)

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2. Topology and topological defects in spinor BEC
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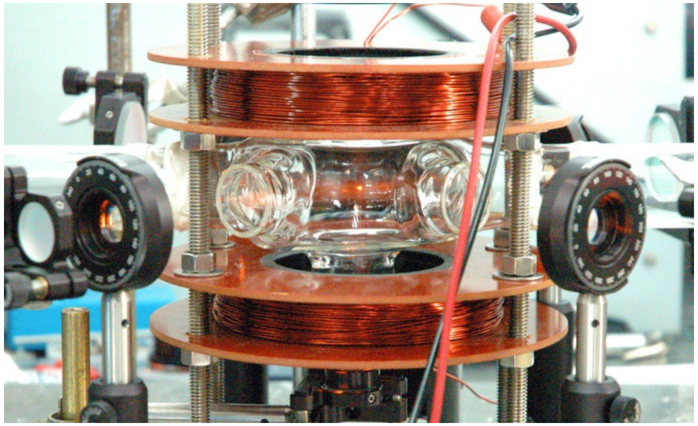
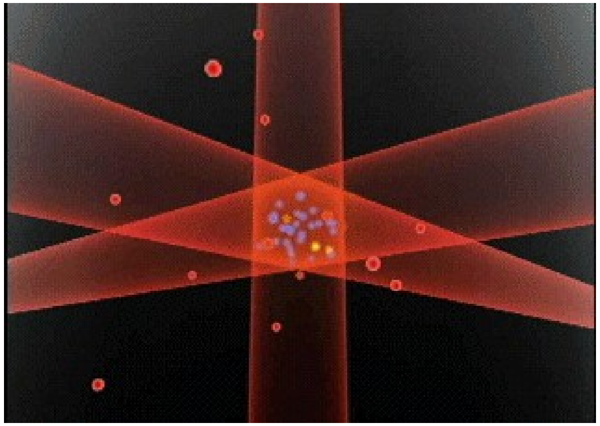
Atomic Bose-Einstein condensates

Dilute alkali atomic Bose-Einstein condensates has been realized in 1997

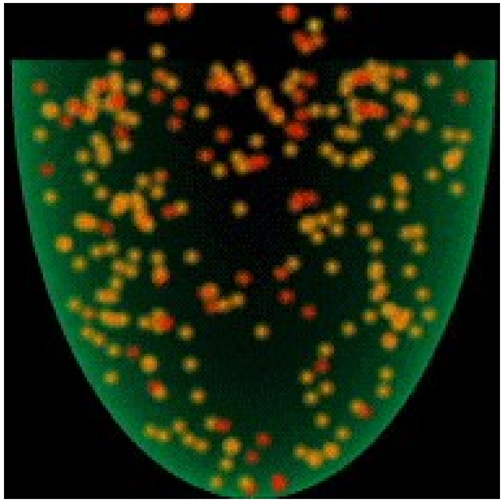


Trap of atoms



Laser cooling of atoms

^{87}Rb, ^{23}Na, ^{7}Li, ^{1}H, ^{85}Rb, ^{41}K, ^{4}He, ^{133}Cs, ^{174}Yb, ^{52}Cr, ^{40}Ca, ^{84}Sr



Evaporating cooling of atoms

Hamiltonian and Gross-Pitaevskii equation

Mean-field Hamiltonian at the zero temperature

$$\mathcal{H} = \int d\boldsymbol{x} \left[\frac{\hbar^2}{2M} |\nabla \Psi|^2 + \frac{c_0}{2} |\Psi|^4 \right]$$

Time evolution of BEC : Gross-Pitaevskii equation

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{\delta \mathcal{H}}{\delta \Psi} = \left[-\frac{\hbar^2}{2M} \nabla^2 + c_0 |\Psi|^2 \right] \Psi$$

$\Psi = |\Psi| \exp[\mathrm{i}\phi]$

$\rho = |\Psi|^2$: Density of BEC

$\boldsymbol{v} = \frac{\hbar}{m} \nabla \phi$: Velocity field of BEC

$U(1)$ gauge symmetry breaking in BEC

$$\mathcal{H} = \int dx \left[\frac{\hbar^2}{2M} |\nabla \Psi|^2 + \frac{c_0}{2} |\Psi|^4 \right]$$

Invariant under $U(1)$ gauge transformation : $\Psi \longrightarrow e^{i\phi}\,\Psi$: symmetry G

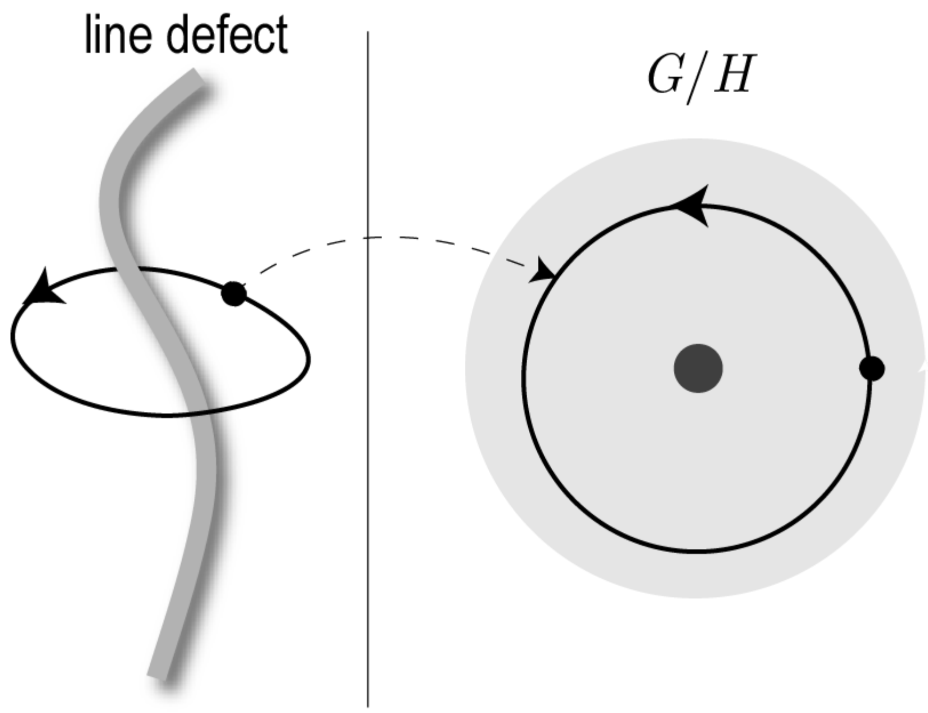
In Bose condensed system, phase ϕ is fixed to one value : Symmetry $H = 1$

Order-parameter manifold (degrees of freedom of Ψ)

$$G/H \cong U(1) \cong S^1$$

Topological defects (fundamental group)

Line defect (fundamental group)

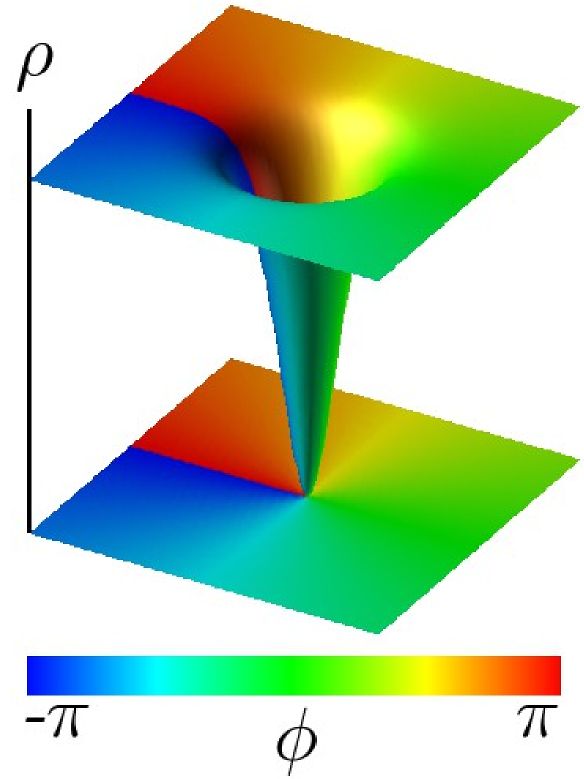


Along the closed path, by how many times the path rotates the singular point of Ψ (fundamental group π_1)

Topological defects in BEC (quantized vortex)

$$G/H \cong U(1) \cong S^1$$

$$\pi_1(S^1) \cong \mathbb{Z}$$ (quantized vortex)



Quantized vortices ($\rho(\boldsymbol{x}) = 0$) around which phase $\phi(\boldsymbol{x})$ changes by integer multiple of 2π : **quantized vortices**

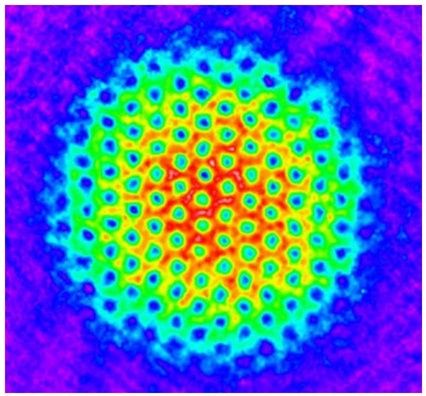
$\Psi = |\Psi| \exp[i\phi]$

$\rho = |\Psi|^2$: Density of BEC

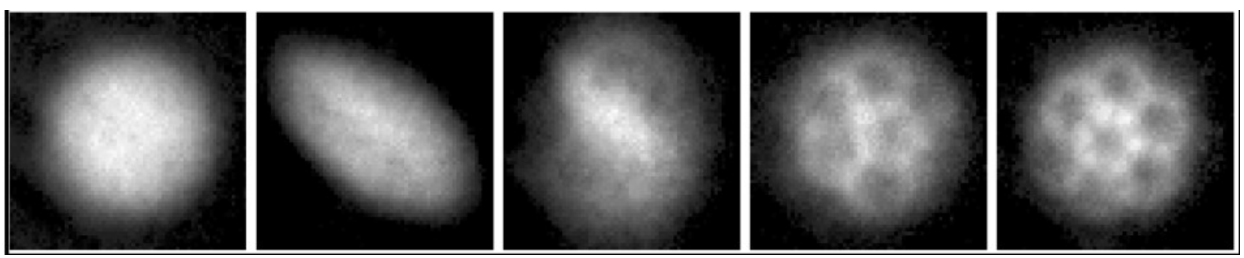
$\boldsymbol{v} = \frac{\hbar}{m}\nabla\phi$: Velocity field of BEC

Experimental observation of vortices

Vortex lattice and its formation in atomic BEC



Vortex lattice in ^{87}Rb BEC



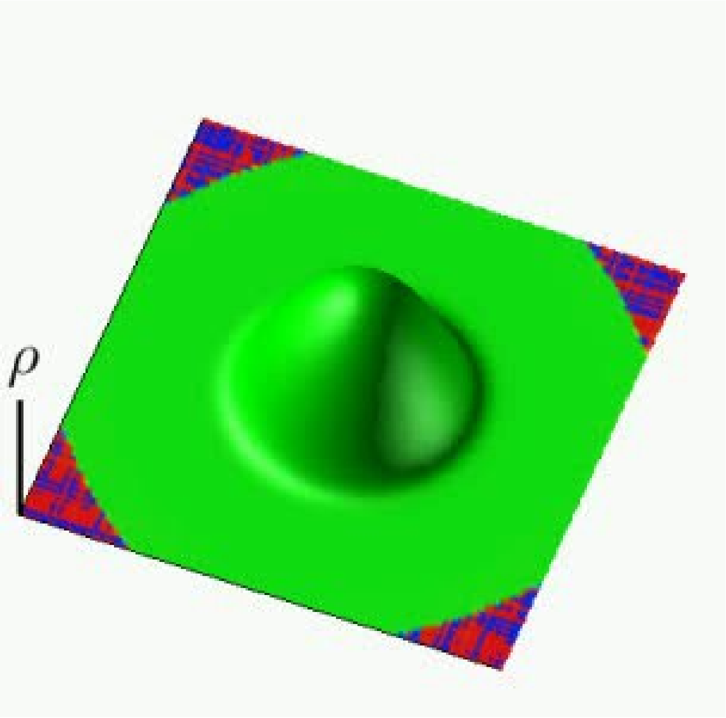
K. W. Madison et al. PRL **86**, 4443 (2001)

Numerical simulation of vortex lattice formation

2D GP equation (in non-dimensional form)

$$(i-\gamma)\frac{\partial \Psi}{\partial t} = \left[-\frac{\nabla^2}{2} - \mu + \frac{\omega^2 \boldsymbol{x}^2}{2} + c_0|\Psi|^2 + i\Omega_z \boldsymbol{x} \times \nabla\right]\Psi$$

K. Kasamatsu et al. PRA **67**, 033610 (2003)



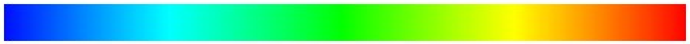
γ : dissipation term

μ : chemical potential

ω : harmonic trap potential

Ω_z : external rotation

$t = 0$: Stationary state with $\Omega_z = 0$

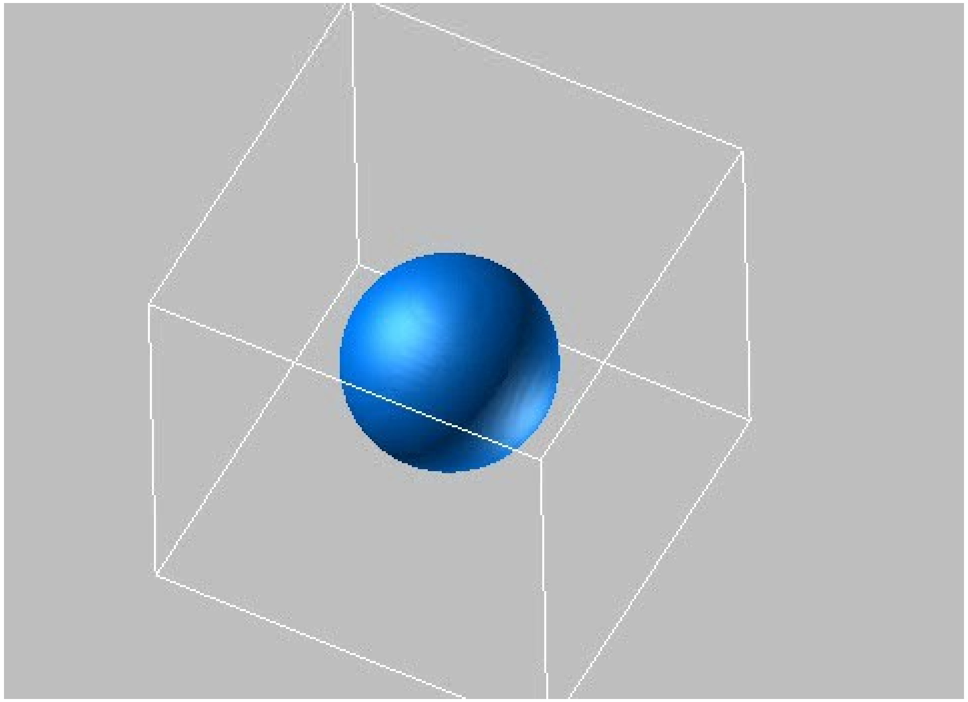


Numerical simulation of vortex lattice formation

3D GP equation (in non-dimensional form)

$$(i-\gamma)\frac{\partial\Psi}{\partial t}=\left[-\frac{\nabla^2}{2}-\mu+\frac{\omega^2\boldsymbol{x}^2}{2}+c_0|\Psi|^2+i\Omega_z(\boldsymbol{x}\times\nabla)_z\right]\Psi$$

K. Kasamatsu et al. PRA **71**, 063616 (2005)

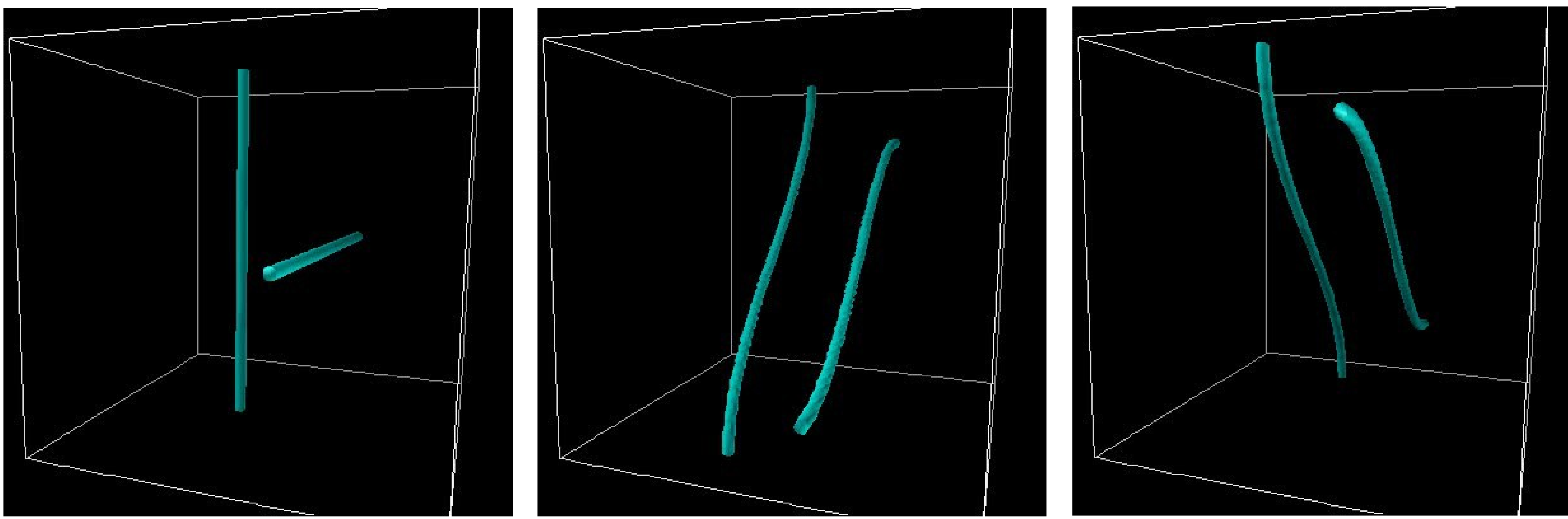


Vortex as line defect

Reconnection of vortices

$$i\frac{\partial \Psi}{\partial t} = \left[-\frac{\nabla^2}{2} + c_0|\Psi|^2\right]\Psi$$

Neumann boundary condition



BEC with larger manifold : 2 - component BEC

Different atom species : ^{87}Rb - ^{41}K

Mixture of Isotope : ^{87}Rb - ^{85}Rb

Different hyperfine state : ^{87}Rb ($F = 1$, $m_F = -1$) - ^{87}Rb ($F = 2$, $m_F = 1$)

$$\mathcal{H} = \int d\boldsymbol{x} \sum_{i=0}^{1} \left[\frac{\hbar^2}{2M} |\nabla \Psi_i|^2 + \frac{c_0}{2} |\Psi_i|^4 + \frac{c_1}{2} |\Psi_i|^2 |\Psi_{1-i}|^2 \right]$$

$$i\hbar \frac{\partial \Psi_0}{\partial t} = \left[-\frac{\hbar^2}{2M} \nabla^2 + c_0 |\Psi_0|^2 + c_1 |\Psi_1|^2 \right] \Psi_0$$

$$i\hbar \frac{\partial \Psi_1}{\partial t} = \left[-\frac{\hbar^2}{2M} \nabla^2 + c_0 |\Psi_1|^2 + c_1 |\Psi_0|^2 \right] \Psi_1$$

$$G/H \cong S^1 \times S^1$$

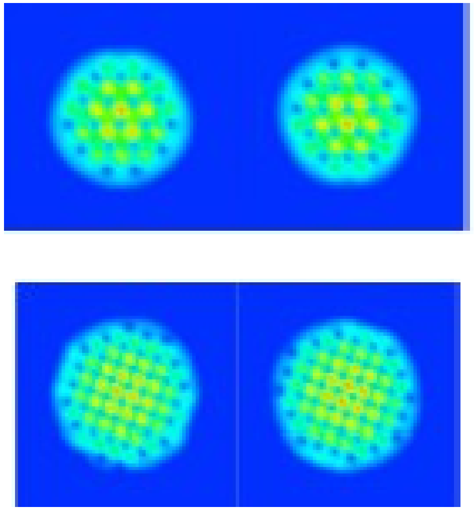
BEC with larger manifold : 2 - component BEC

$$G/H \cong S^1 \times S^1$$

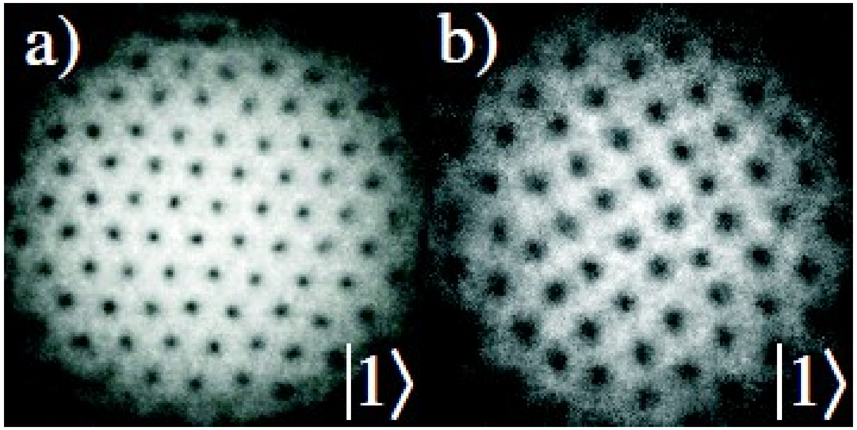
$$i\hbar\frac{\partial\Psi_0}{\partial t} = \left[-\frac{\hbar^2}{2M}\nabla^2 + V + c_0|\Psi_0|^2 + c_1|\Psi_1|^2\right]\Psi_0$$

$$i\hbar\frac{\partial\Psi_1}{\partial t} = \left[-\frac{\hbar^2}{2M}\nabla^2 + V + c_0|\Psi_1|^2 + c_1|\Psi_0|^2\right]\Psi_1$$

K. Kasamatsu et al. PRL **91**, 150406 (2003)



V. Schweikhard et al. PRL **93**, 210403 (2004)

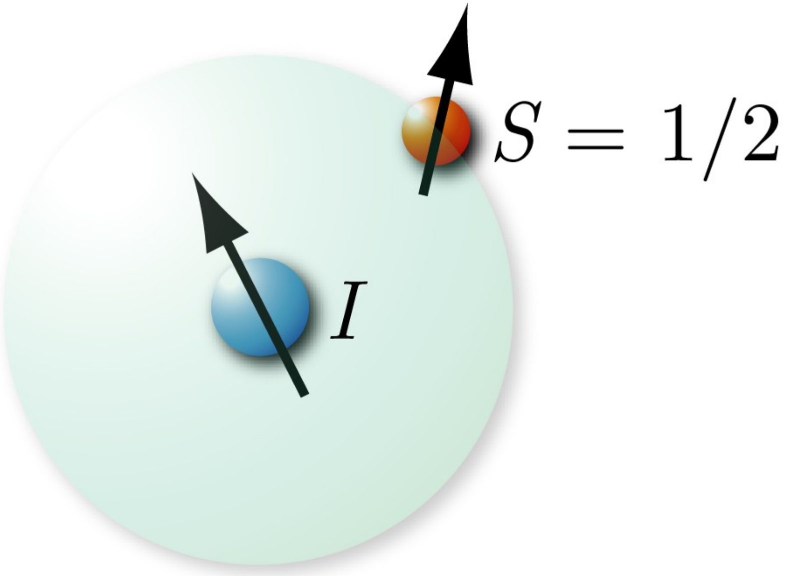


Spinor BEC

BEC with spin degrees of freedom

Hyperfine coupling of electron end nuclear spin

$$(F = I + L + S)$$



^{87}Rb, ^{23}Na, ^{7}Li, ^{41}K	F=1, 2
^{85}Rb	F=2, 3
^{133}Cs	F=3, 4
^{52}Cr	S=3, I=0

Spinor BEC

BEC with spin degrees of freedom

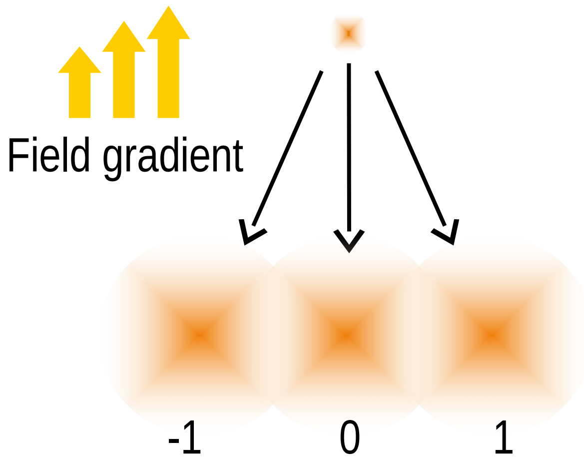
^{87}Rb $(I = 3/2)$

$$F = 2 \begin{cases} m_F = 2 \\ m_F = 1 \\ m_F = 0 \\ m_F = -1 \\ m_F = -2 \end{cases} \qquad F = 1 \begin{cases} m_F = 1 \\ m_F = 0 \\ m_F = -1 \end{cases}$$

Spin 1 : 3-component BEC

$$\Psi = (\psi_1, \psi_0, \psi_{-1})$$

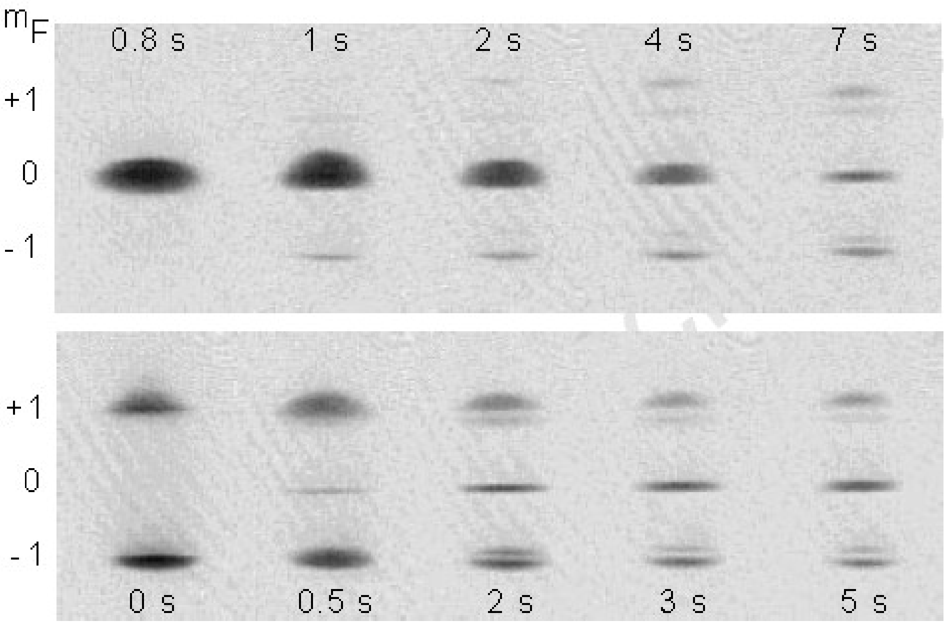
Multicomponent BEC labeled by magnetic sublevel m_F



Spinor BEC

Stern-Gerlach experiment

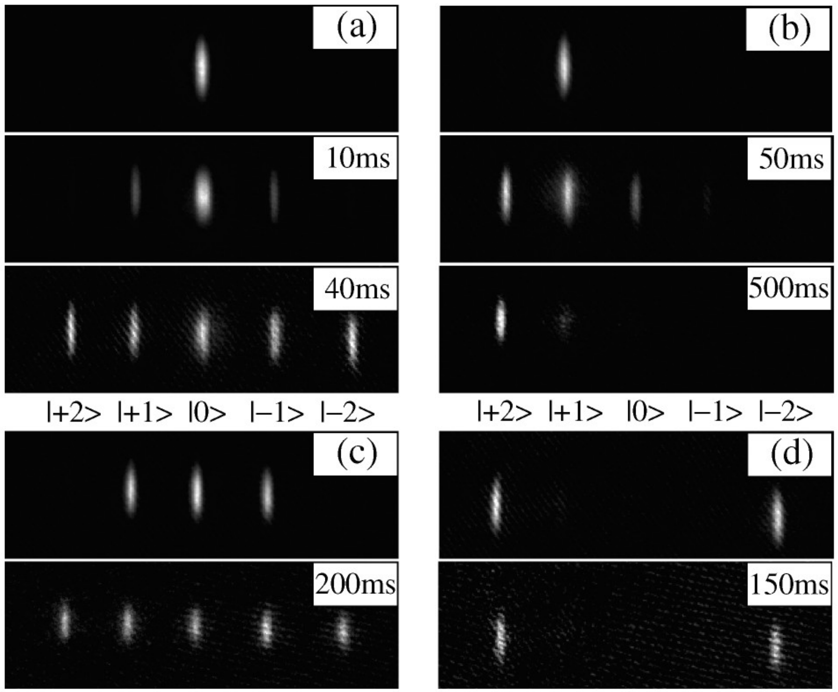
$F = 1$



J. Stenger et al. Nature **396**, 345 (1998)

Rotation of Spin can be observed

$F = 2$



H. Schmaljohann et al. PRL **92**, 040402 (2004)

Theory of Spinor BEC

Hamiltonian of spinor Bosons

$$H = -\int d\boldsymbol{x}\, \frac{\hbar^2}{2M} \nabla \Psi^\dagger_m(\boldsymbol{x}) \nabla \Psi_m(\boldsymbol{x})$$
$$+ \frac{1}{2} \int d\boldsymbol{x}_1 \int d\boldsymbol{x}_2 \Psi^\dagger_{m_1}(\boldsymbol{x}_1) \Psi^\dagger_{m_2}(\boldsymbol{x}_2) V_{m_1 m_2 m_1' m_x'}(\boldsymbol{x}_1 - \boldsymbol{x}_2) \Psi_{m_2'}(\boldsymbol{x}_2) \Psi_{m_1'}(\boldsymbol{x}_1)$$

Low energy contact interaction $(l = 0)$

$$V_{m_1 m_2 m_1' m_x'}(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \delta(\boldsymbol{x}_1 - \boldsymbol{x}_2) \sum_{F=0,2,4} g_F \sum_{m_1, m_2, m_1', m_2', M} O^{F,M}_{m_1 m_2} \left(O^{F,M}_{m_1' m_2'} \right)^*$$

Mean-field Hamiltonian (spin-1)

$$H = \int d\boldsymbol{x} \left[\frac{\hbar^2}{2M} \sum_{m=-1}^{1} \nabla \Psi_m^* \nabla \Psi_m + \frac{c_0}{2}\rho^2 + \frac{c_1}{2}\boldsymbol{F}^2 \right]$$

Density Spin

$$F_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad F_+ = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$F_- = F_+^T, \quad F_x = \frac{F_+ + F_-}{2}, \quad F_y = \frac{F_+ - F_-}{2i}$$

$$\rho = \sum_{m=-1}^{1} |\Psi_m|^2$$

$$\boldsymbol{F} = (\Psi_1^* \ \Psi_0^* \ \Psi_{-1}^*)(F_x, F_y, F_z) \begin{pmatrix} \Psi_1 \\ \Psi_0 \\ \Psi_{-1} \end{pmatrix}$$

Gauge and spin rotation symmetry of wave function are broken

$$\Psi' = e^{i\phi} \, e^{-i\boldsymbol{n}\cdot\boldsymbol{F}\alpha} \, \Psi \ \ (G = U(1)_{\boldsymbol{F}} \times SO(3)_{\phi})$$

Possible phase

$$H = \int d\boldsymbol{x} \left[\frac{\hbar^2}{2M} \sum_{m=-1}^{1} \nabla\Psi_m^* \nabla\Psi_m + \frac{c_0}{2}\rho^2 + \frac{c_1}{2}\boldsymbol{F}^2 \right]$$

$c_1 > 0$: polar (^{23}Na BEC)

$$e^{i\phi} e^{-i\boldsymbol{n}\cdot\hat{\boldsymbol{F}}\alpha} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\boldsymbol{F} = 0$$

$$\frac{G}{H} \simeq \frac{U(1)_\phi \times S_F^2}{(\mathbb{Z}_2)_{\phi+F}}$$

$c_1 < 0$: Ferromagnetic (^{87}Rb BEC)

$$e^{i\phi} e^{-i\boldsymbol{n}\cdot\hat{\boldsymbol{F}}\alpha} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\boldsymbol{F} \neq 0$$

$$\frac{G}{H} \simeq SO(3)_{\phi+F}$$

Gross-Pitaevskii equation

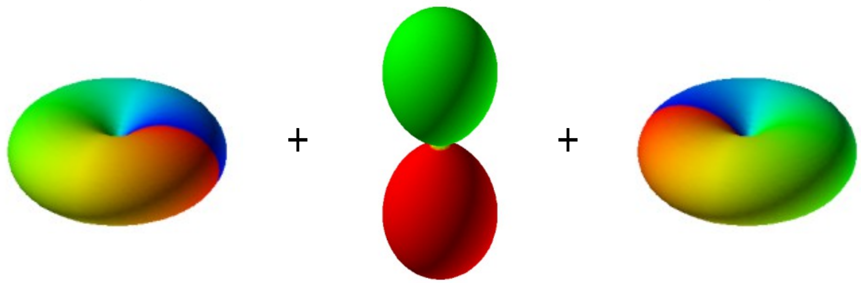
$$i\hbar\frac{\partial\Psi_{\pm1}}{\partial t} = \left(-\frac{\hbar^2}{2M}\nabla^2 + V + c_0\rho\right)\Psi_{\pm1} + c_1\left(\frac{1}{\sqrt{2}}F_{\mp}\Psi_0 \pm F_z\Psi_{\pm}\right)$$

$$i\hbar\frac{\partial\Psi_0}{\partial t} = \left(-\frac{\hbar^2}{2M}\nabla^2 + V + c_0\rho\right)\Psi_0 + \frac{c_1}{\sqrt{2}}\left(F_{+}\Psi_1 + F_{-}\Psi_{-1}\right)$$

Graphical image by the spherical Harmonics

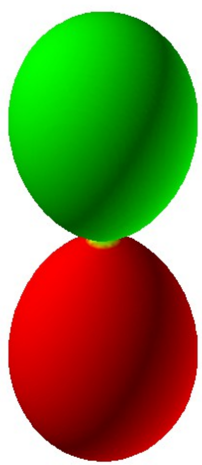
$$\sum_{m=-1}^{1} \Psi_m Y_{2,m}$$

$Y_{1,1}$ $Y_{1,0}$ $Y_{1,-1}$



$\cos\theta$ $-e^{i\varphi}\sin\theta$

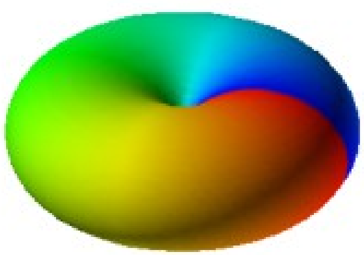
Polar



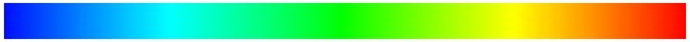
0

π

Ferromagnetic



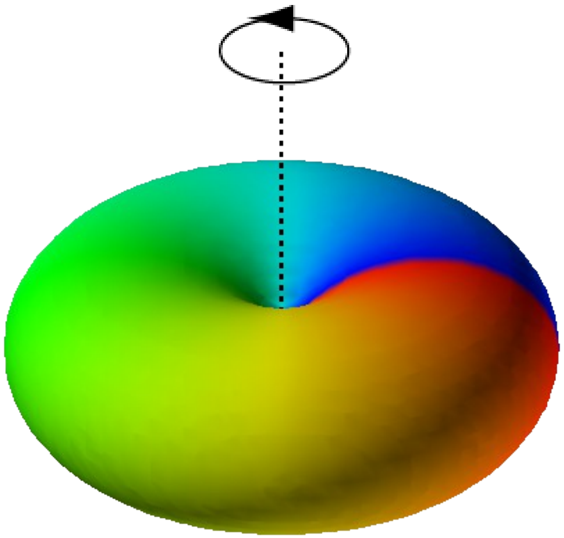
$-\pi$ π



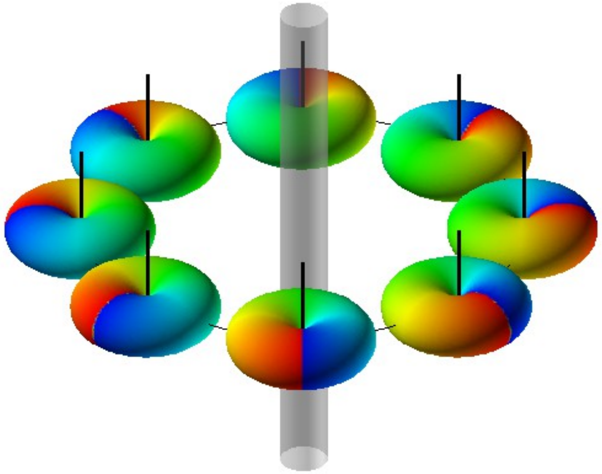
Topological defect in Ferromagnetic state

$$\frac{G}{H} \simeq \frac{U(1)_\phi \times SO(3)_F}{(U(1))_{\phi+F}}$$

$$\simeq SO(3)_{\phi+F}$$

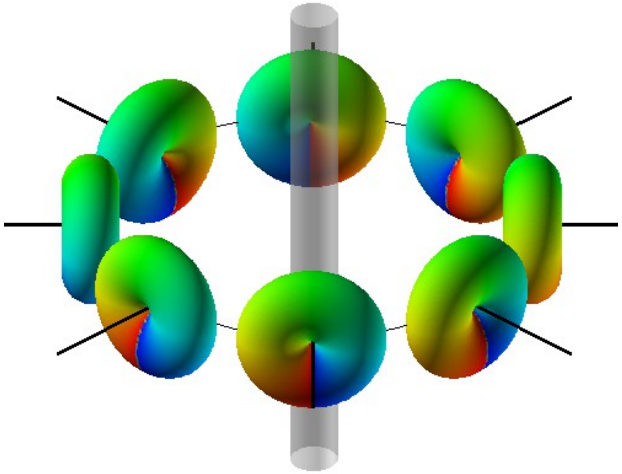


Gauge vortex



Continuous transformation

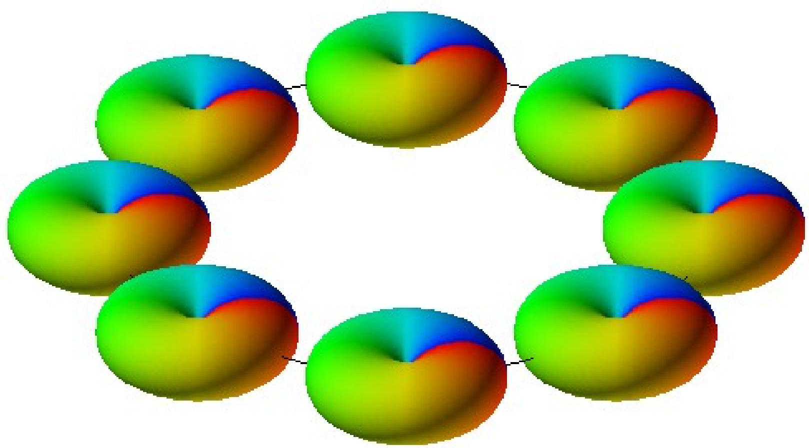
Spin vortex



Topological defect in Ferromagnetic state

$$\pi_1[SO(3)_{\phi+\boldsymbol{F}}] \cong \mathbb{Z}_2$$

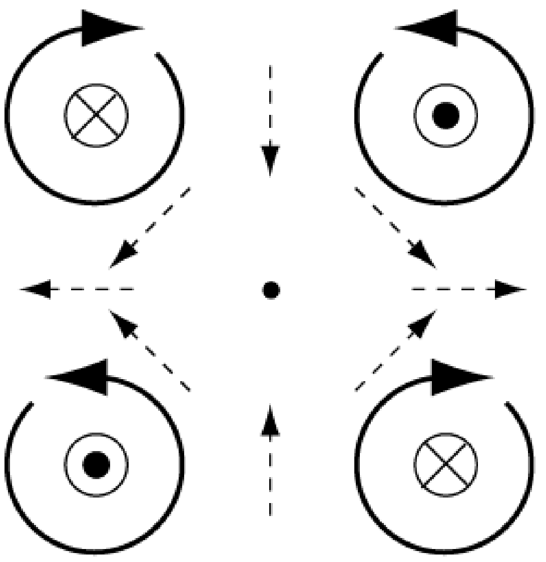
Doubly winding state is no longer topological defect

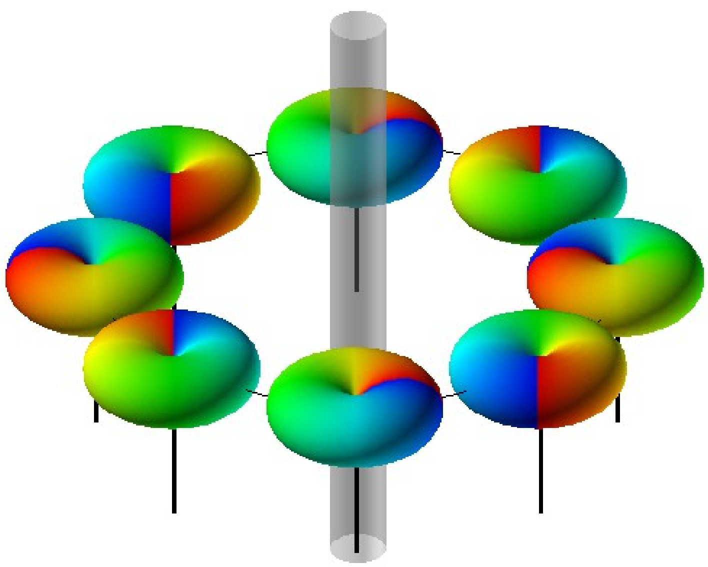


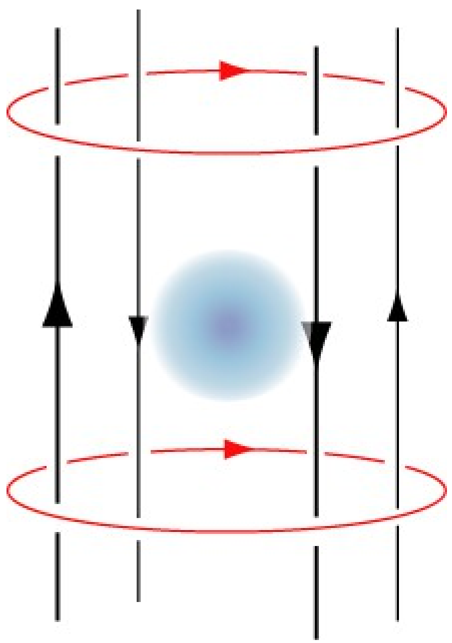
T. Ishoshima, et al. PRA **61**, 063610 (1999)

Topological excitation in Ferromagnetic state

Creation of doubly winding state from zero winding







$$(\Psi_1, (\Psi_1, \Psi_0, \Psi_{-1}) = (0, 0, e^{-2i\varphi})^{-2i\varphi}/2)$$

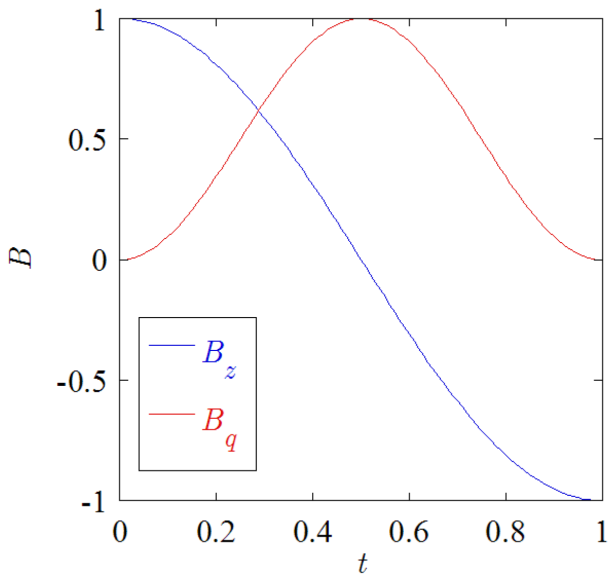
Adiabatic change of quadratic magnetic field

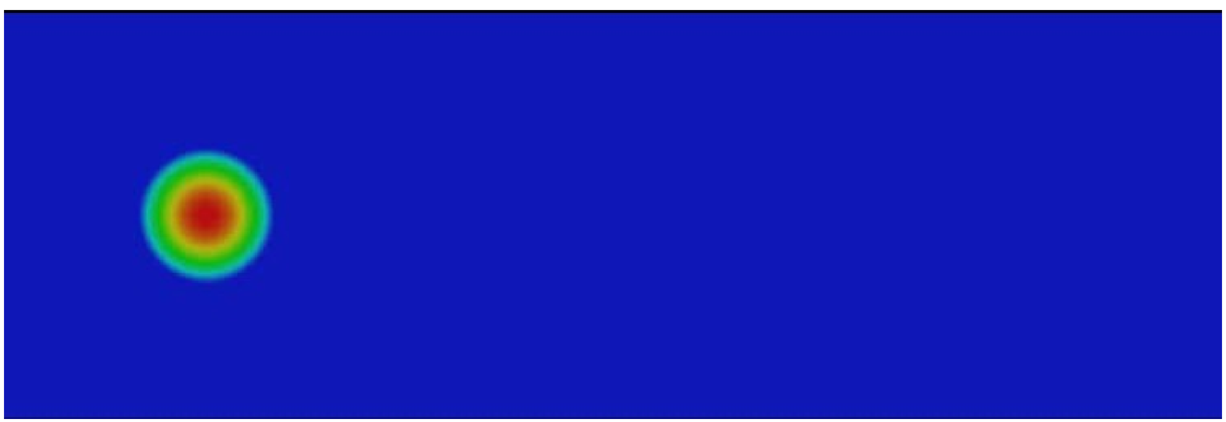
Topological excitation in Ferromagnetic state

Creation of doubly winding state from zero winding

$$i\hbar\frac{\partial\Psi_m}{\partial t} = \left(-\frac{\hbar^2}{2M}\nabla^2 + V + B\right)\Psi_m + \text{interaction}$$

$$B = \begin{pmatrix} B_z & B_q e^{i\varphi}/\sqrt{2} & 0 \\ B_q e^{-i\varphi}/\sqrt{2} & 0 & B_q e^{i\varphi}/\sqrt{2} \\ 0 & B_q e^{-i\varphi}/\sqrt{2}/\sqrt{2} & -B_z \end{pmatrix}$$



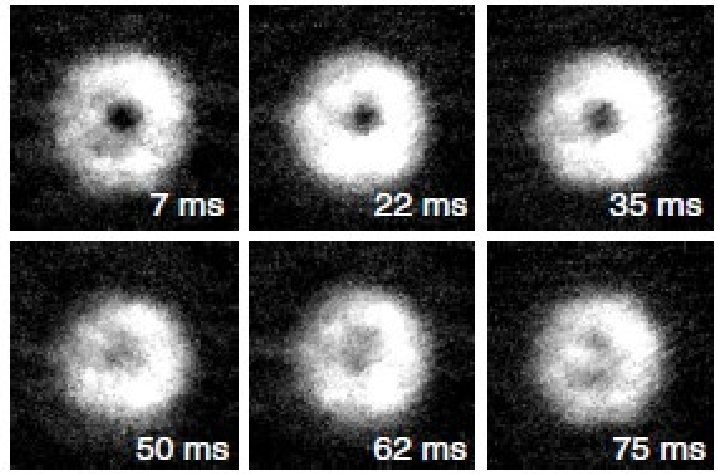


Topological excitation in Ferromagnetic state

Creation of doubly winding state from zero winding

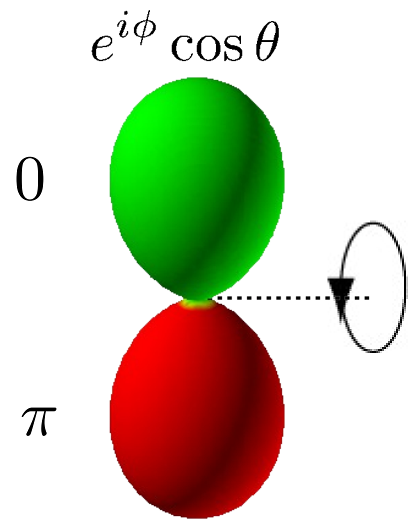
$$i\hbar \frac{\partial \Psi_m}{\partial t} = \left(-\frac{\hbar^2}{2M} \nabla^2 + V + B \right) \Psi_m + \text{interaction}$$

$$B = \begin{pmatrix} B_z & B_q e^{i\varphi}/\sqrt{2} & 0 \\ B_q e^{-i\varphi}/\sqrt{2} & 0 & B_q e^{i\varphi}/\sqrt{2} \\ 0 & B_q e^{-i\varphi}/\sqrt{2}/\sqrt{2} & -B_z \end{pmatrix}$$



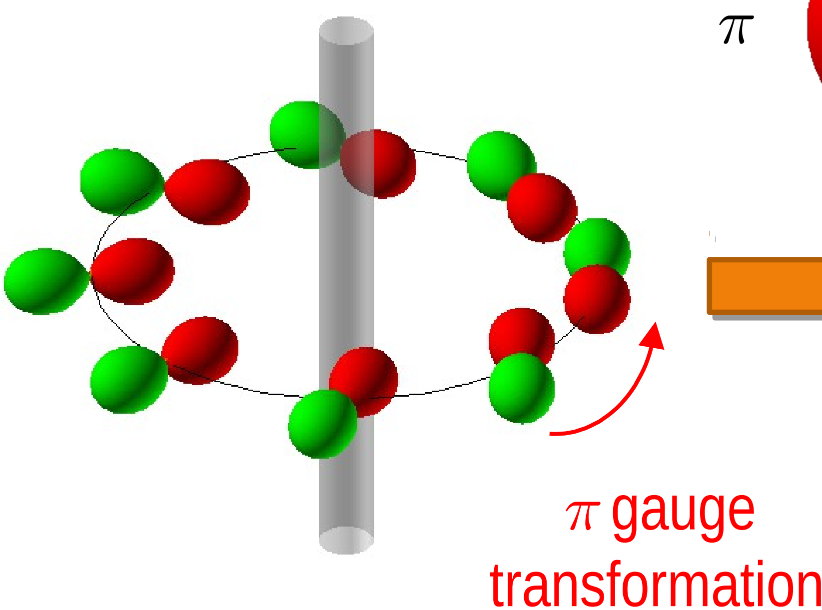
Y. Shin, et al. PRL **93**, 160406 (2004)

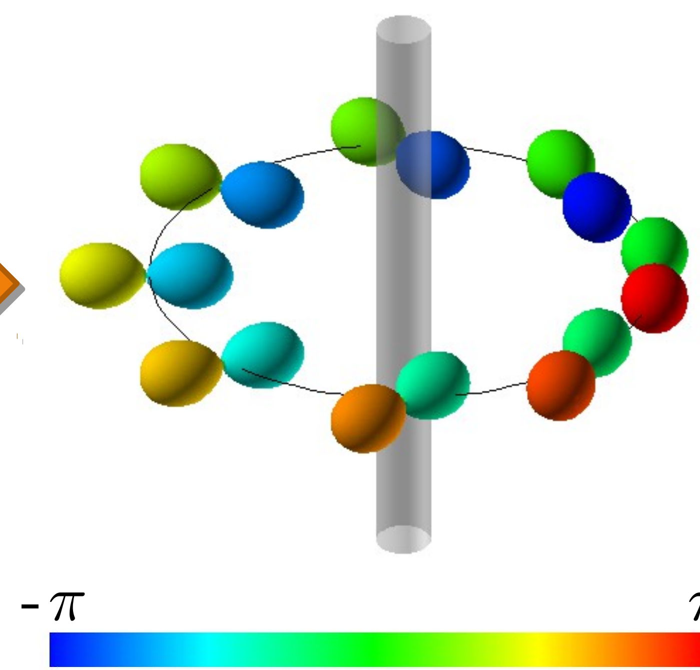
Topological defect in polar state



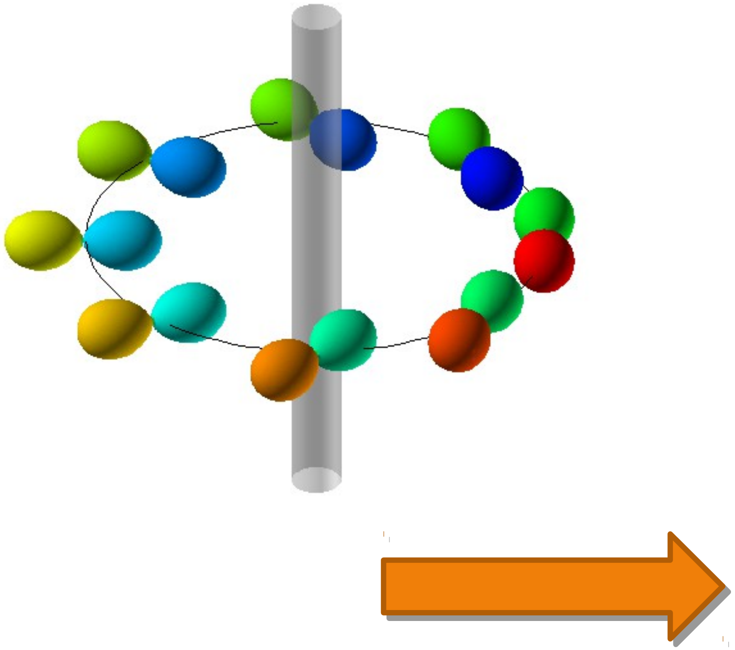
$$\frac{G}{H} \simeq \frac{U(1)_\phi \times SO(3)_F}{SO(2)_F \rtimes (\mathbb{Z}_2)_{\phi+F}} \simeq \frac{U(1)_\phi \times S^2_F}{(\mathbb{Z}_2)_{\phi+F}}$$

$$\pi_1[G/H] \simeq \mathbb{Z}$$

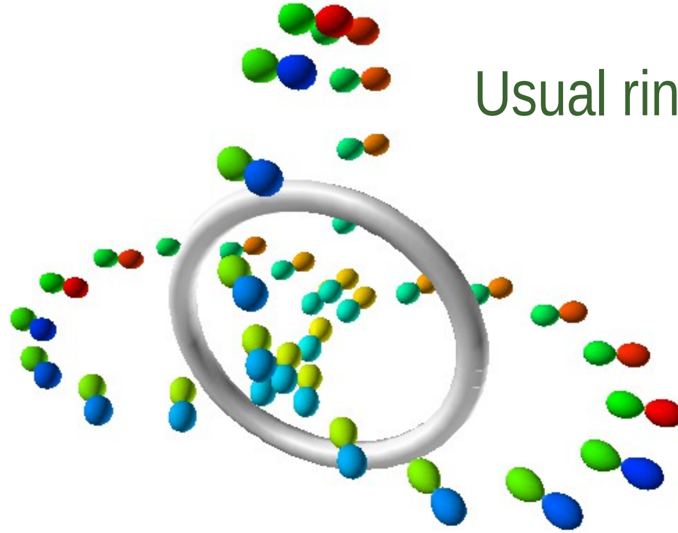




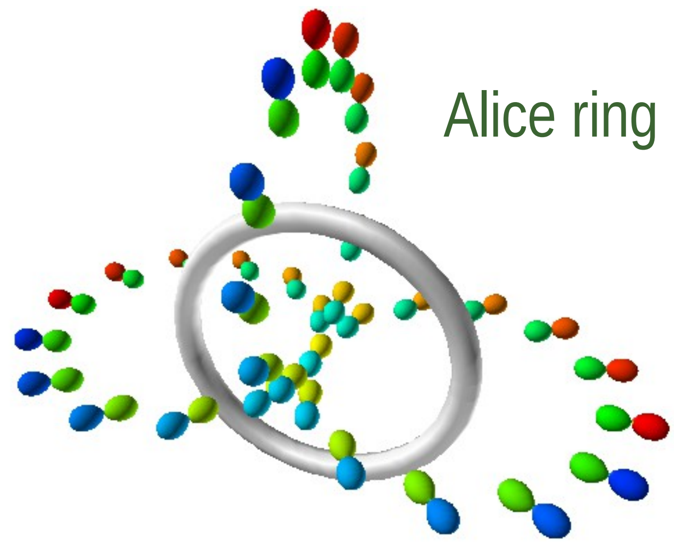
Vortex ring in polar phase



Vortex ring



Usual ring



Alice ring

Spin-2 case

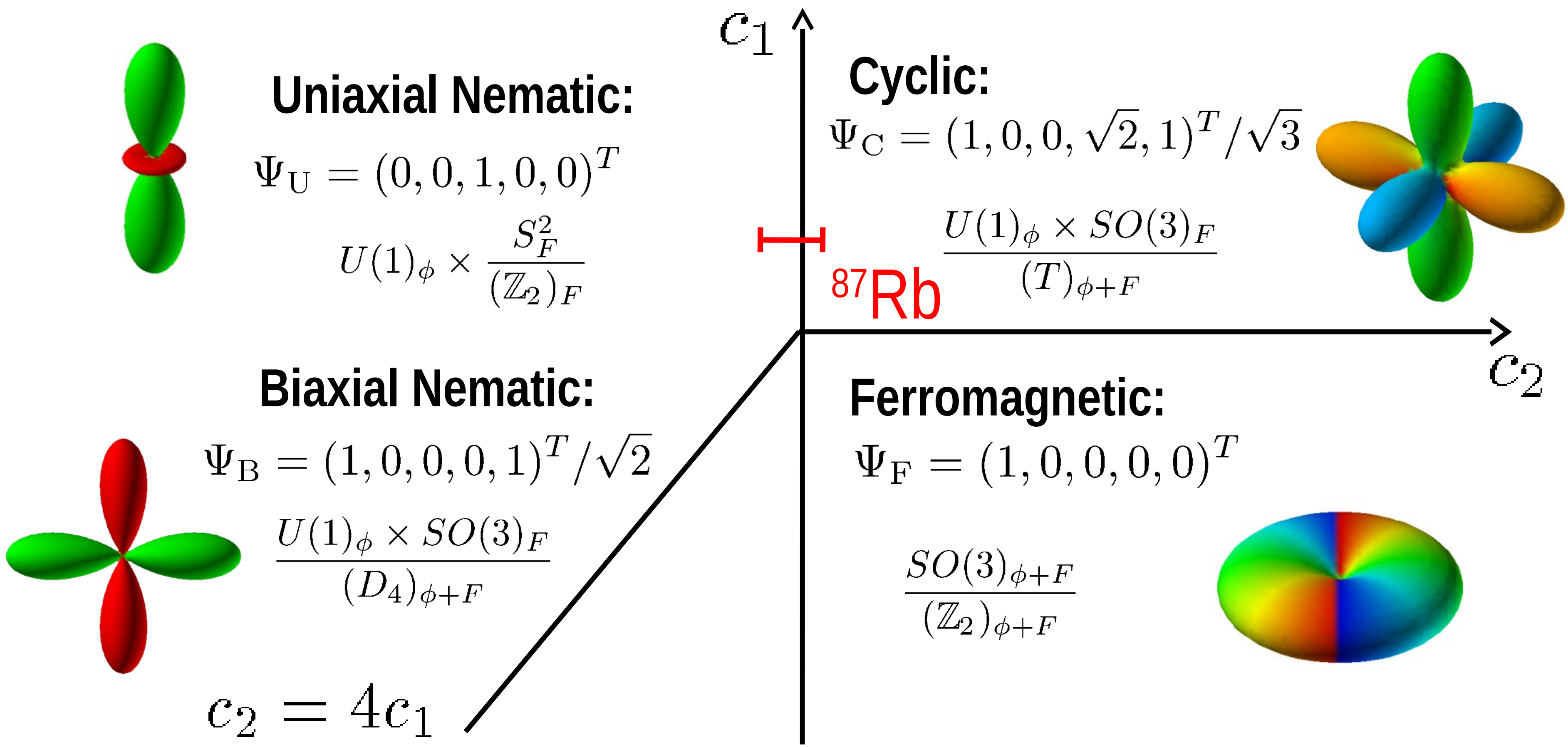
$$H = \int d\boldsymbol{x} \left[-\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + \frac{c_0}{2}\rho^2 + \frac{c_1}{2}\boldsymbol{F}^2 + \frac{c_2}{2}|A_{20}|^2 \right]$$

$$A_{00}(\boldsymbol{x}) = 2\Psi_2(\boldsymbol{x})\Psi_{-2}(\boldsymbol{x}) - 2\Psi_1(\boldsymbol{x})\Psi_{-1}(\boldsymbol{x}) + \Psi_0(\boldsymbol{x})^2$$

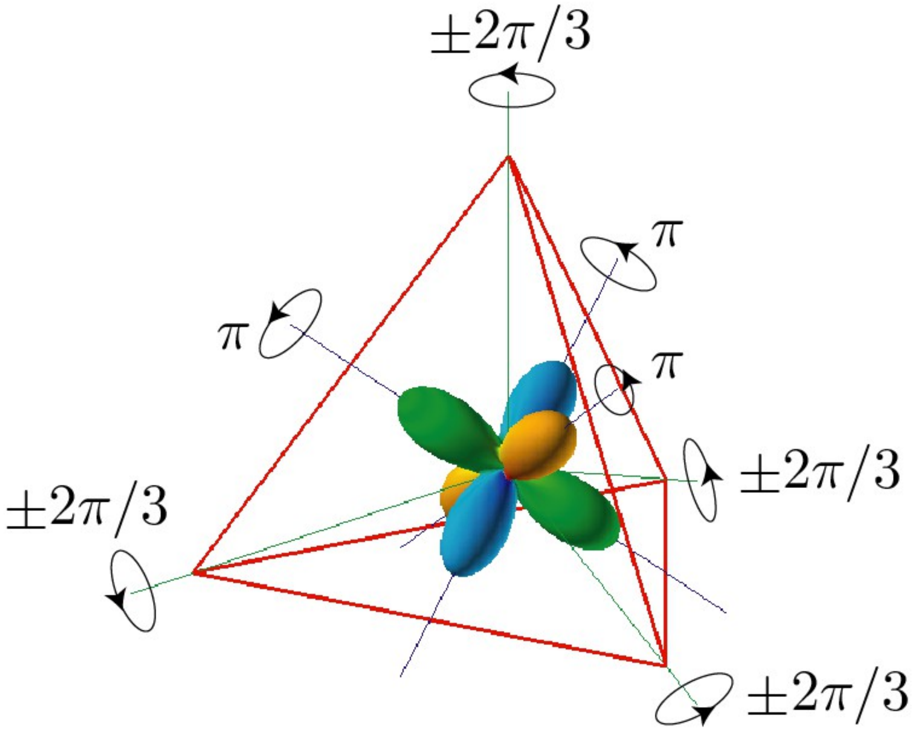
Singlet-pair amplitude

Spin-2 case

$$H = \int dx \left[-\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + \frac{c_0}{2}\rho^2 + \frac{c_1}{2}\boldsymbol{F}^2 + \frac{c_2}{2}|A_{20}|^2 \right]$$



Topological defect in cyclic state



$$\frac{U(1)_\phi \times SO(3)_F}{(T)_{\phi+F}}$$

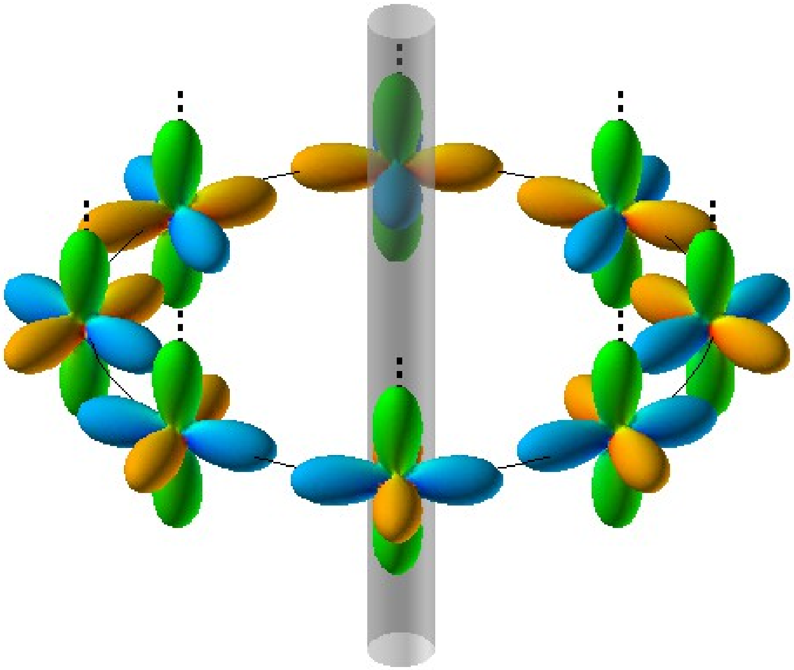
Topological defects can be labeled by 12 rotations keeping tetrahedron invariant

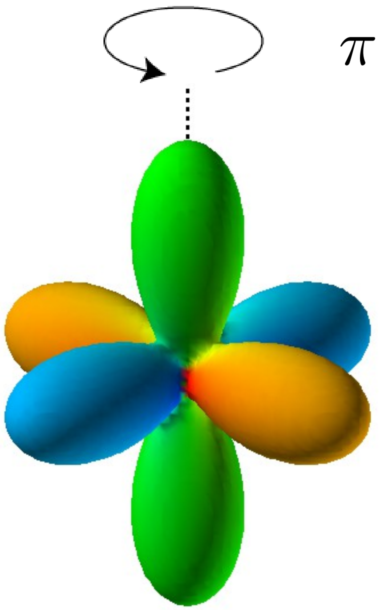
↓

Non-Abelian topological defect!

Topological excitation in cyclic state

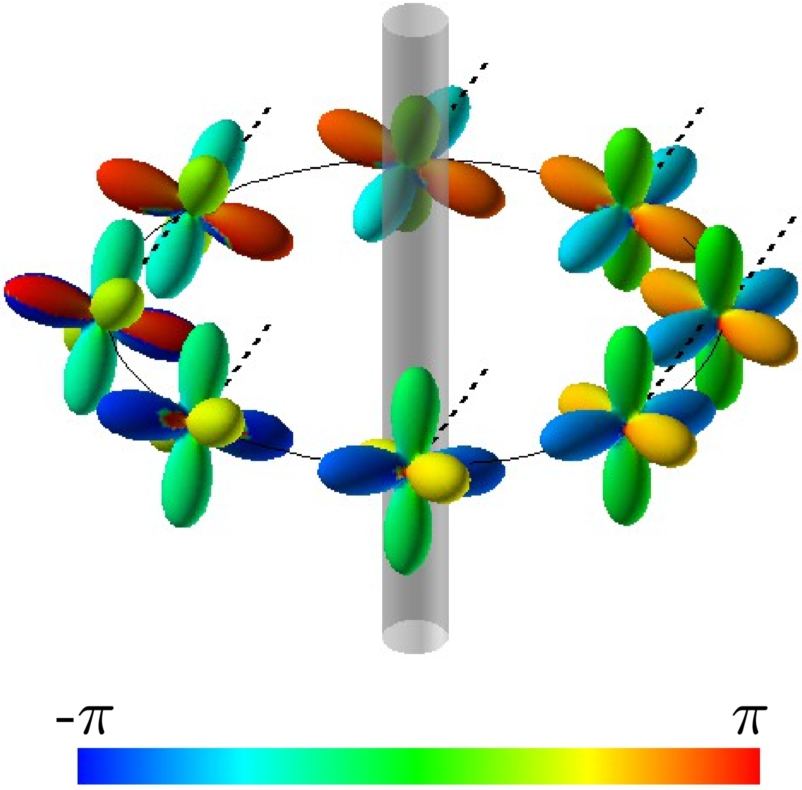
1/2-spin vortex

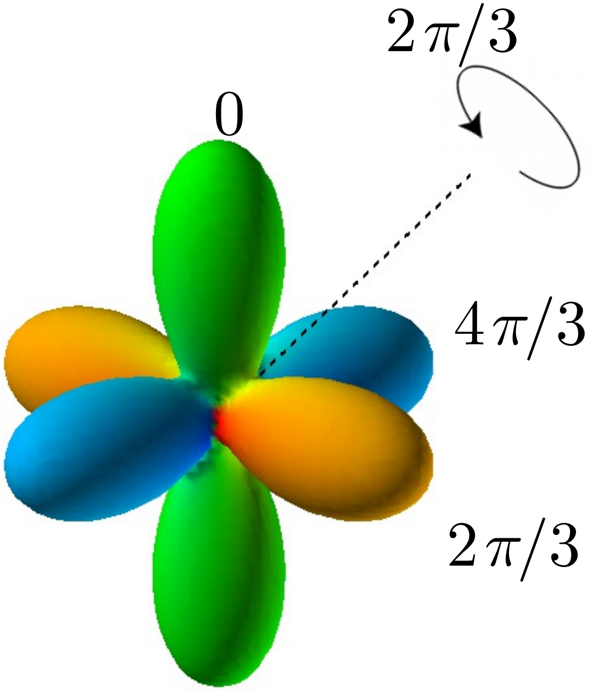




Topological excitation in cyclic state

1/3 vortex





Gross-Pitaevskii Equation

$$i\hbar\frac{\partial\Psi_2}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\Psi_2 + c_0\rho\Psi_2 + c_1(F_-\Psi_1 + 2F_z\Psi_2) + c_2A_{20}\Psi^*_{-2}$$

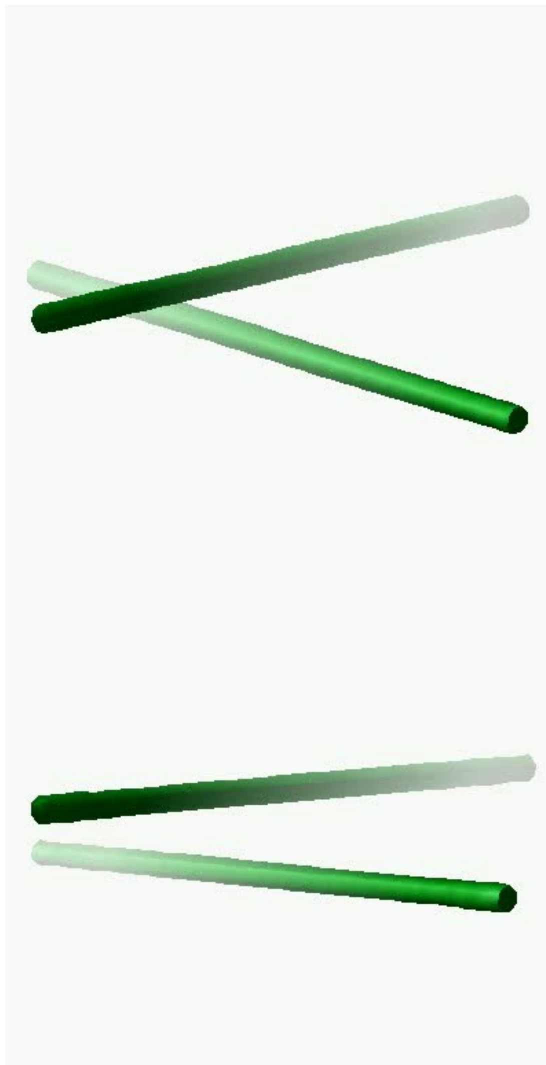
$$i\hbar\frac{\partial\Psi_1}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\Psi_1 + c_0\rho\Psi_1 + c_1\left(\frac{\sqrt{6}}{2}F_-\Psi_0 + F_+\Psi_2 + F_z\Psi_1\right) - c_2A_{20}\Psi^*_{-1}$$

$$i\hbar\frac{\partial\Psi_0}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\Psi_0 + c_0\rho\Psi_0 + \frac{\sqrt{6}}{2}c_1(F_-\Psi_{-1} + F_+\Psi_1) + c_2A_{20}\Psi^*_0$$

$$i\hbar\frac{\partial\Psi_{-1}}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\Psi_{-1} + c_0\rho\Psi_{-1} + c_1\left(\frac{\sqrt{6}}{2}F_+\Psi_0 + F_-\Psi_{-2} - F_z\Psi_{-1}\right) - c_2A_{20}\Psi^*_1$$

$$i\hbar\frac{\partial\Psi_{-2}}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\Psi_{-2} + c_0\rho\Psi_{-2} + c_1(F_+\Psi_{-1} - 2F_z\Psi_{-2}) + c_2A_{20}\Psi^*_2$$

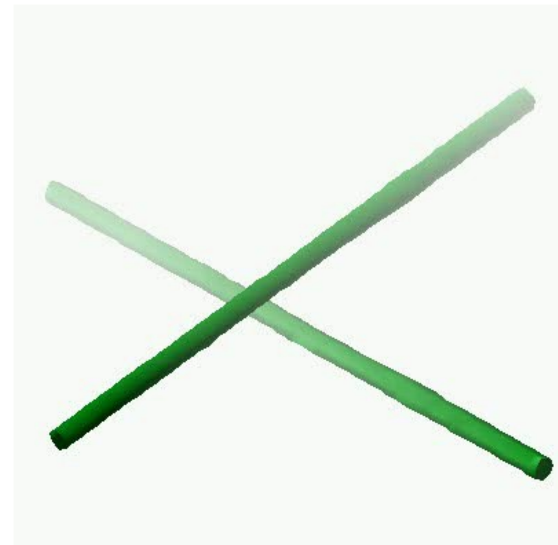
Collision of vortices with non-commutative charge forms a new “**rung**” vortex connecting two vortices



Reconnection

Abelian excitation

Passing



Non-Abelian excitation

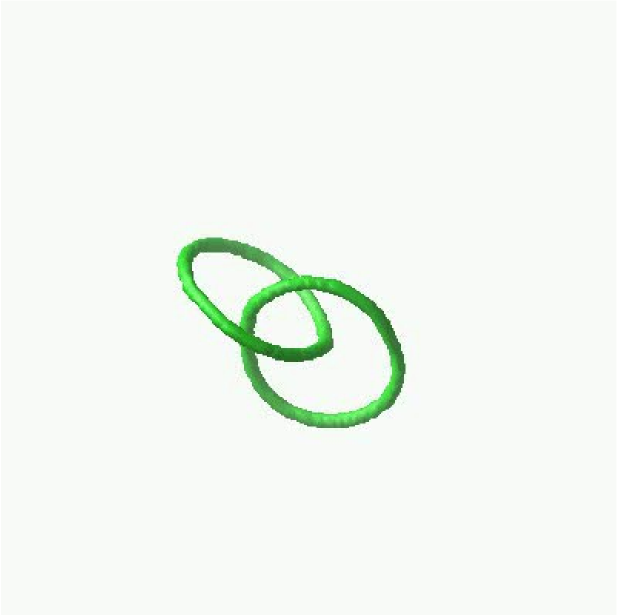
Rung vortex

M. Kobayashi, et al. PRL **103**, 115301(2009)

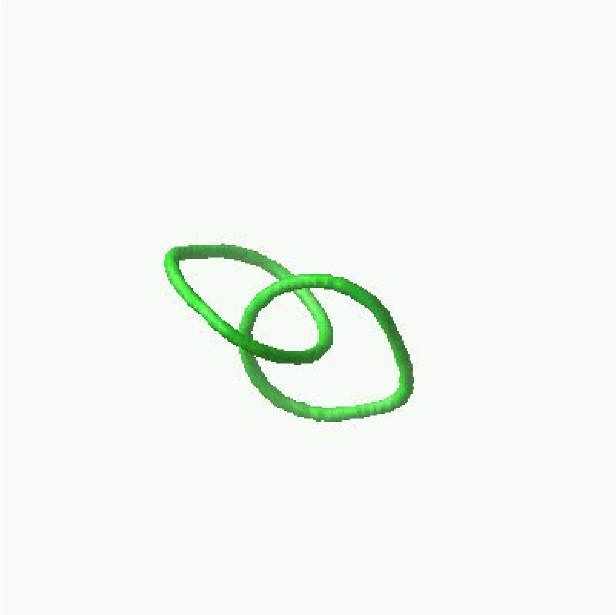
Collision dynamics of topological excitations

Abelian excitation

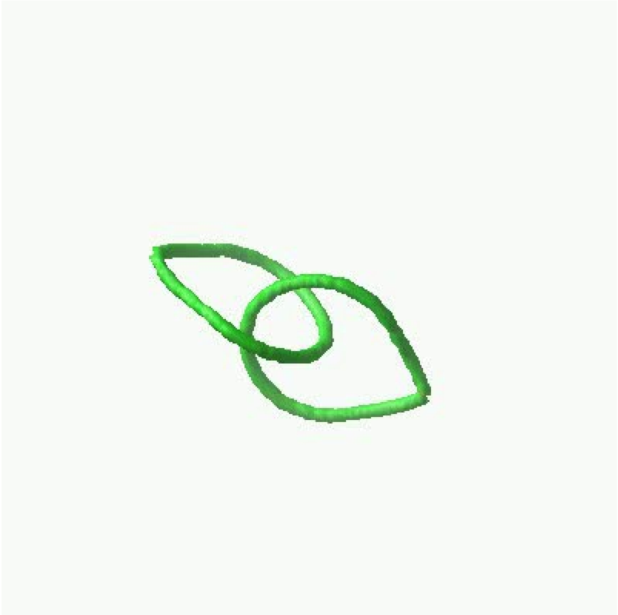
Non-Abelian excitation



Large ring



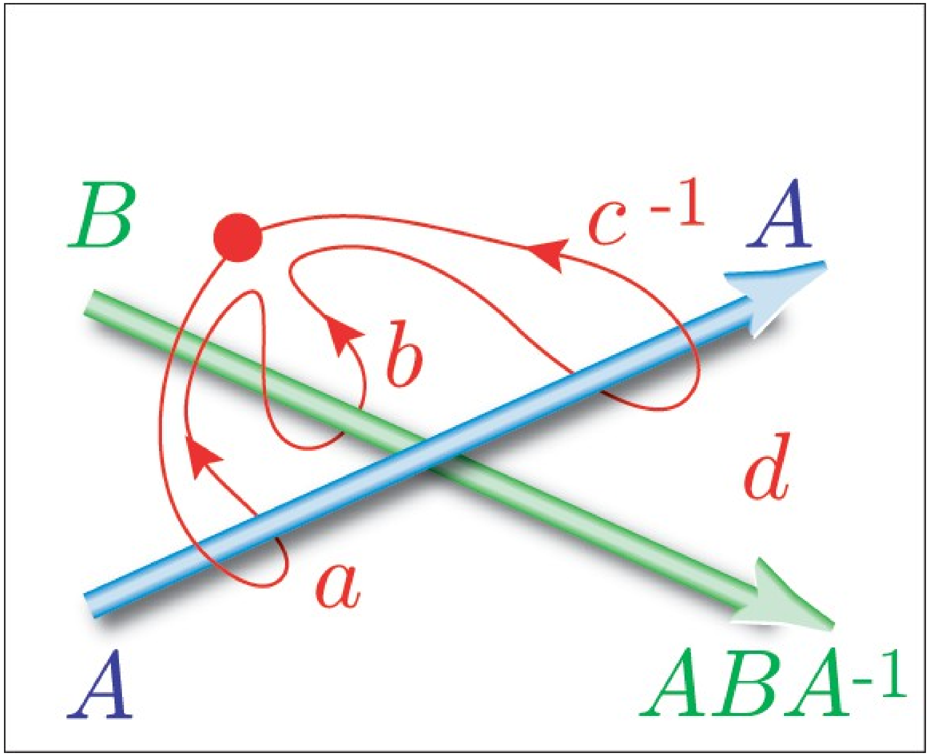
Unraveling of link



New excitation

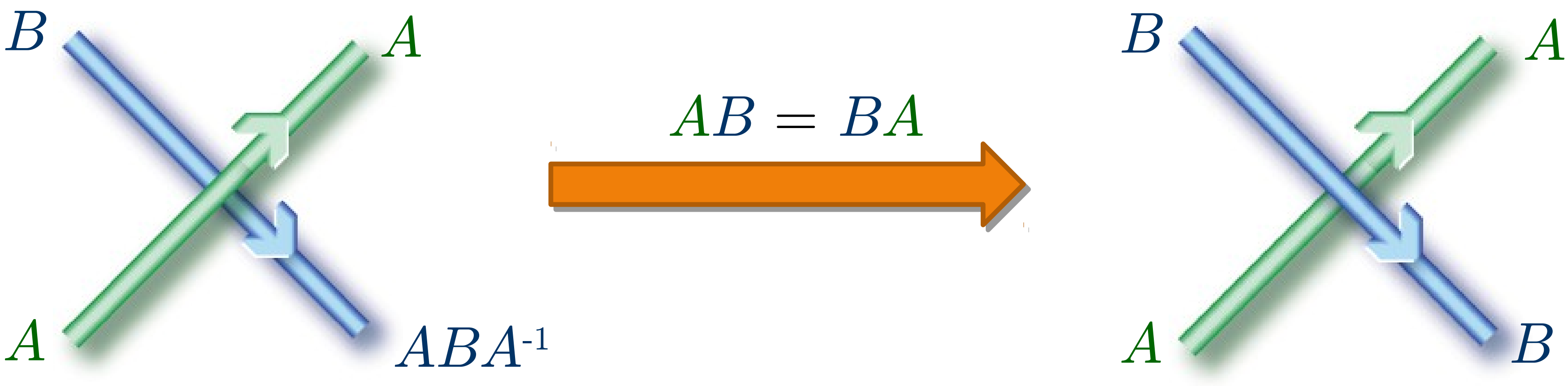
Linked non-Abelian excitations cannot unravel because of the formation of the new excitation.

Topological charge of topological excitation



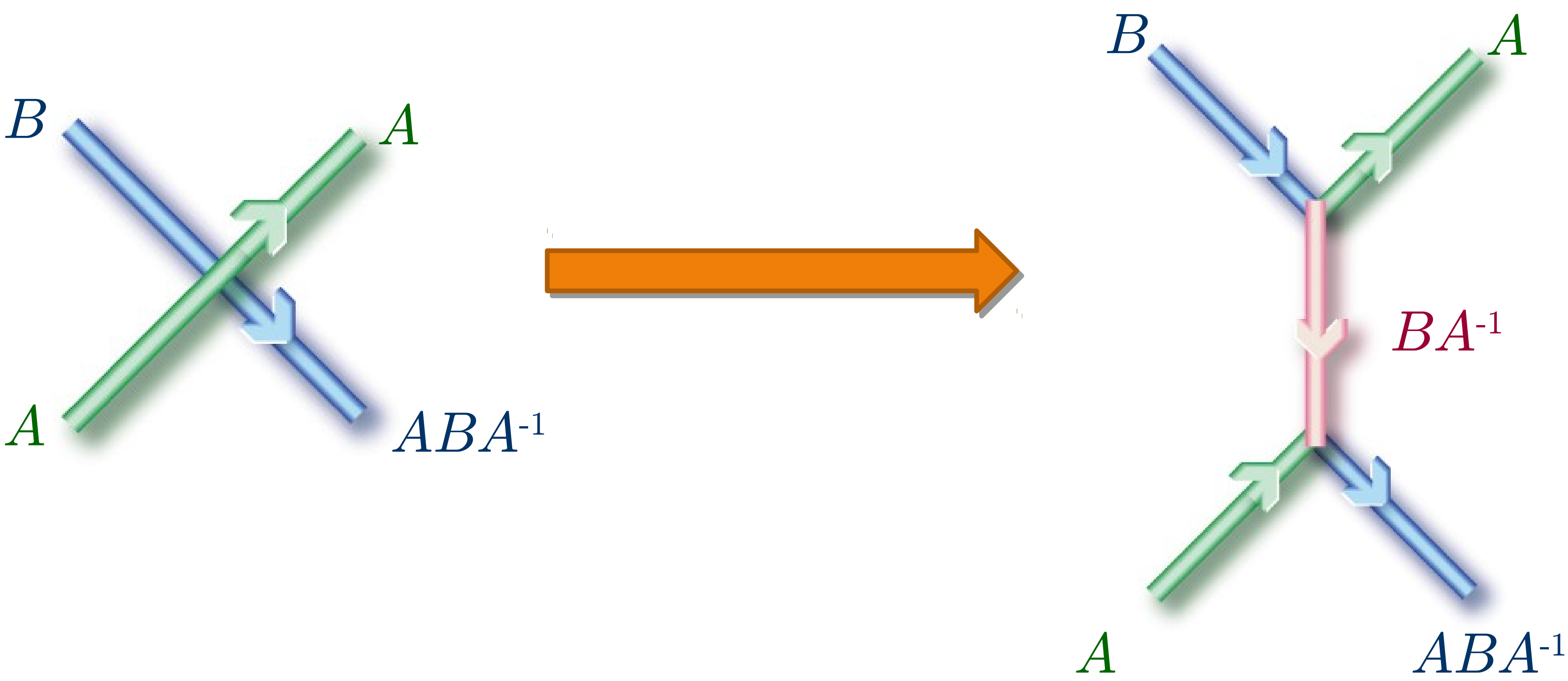
Topological invariant of excitations can be fixed by a closed path encircling the excitations

Collision of Vortex



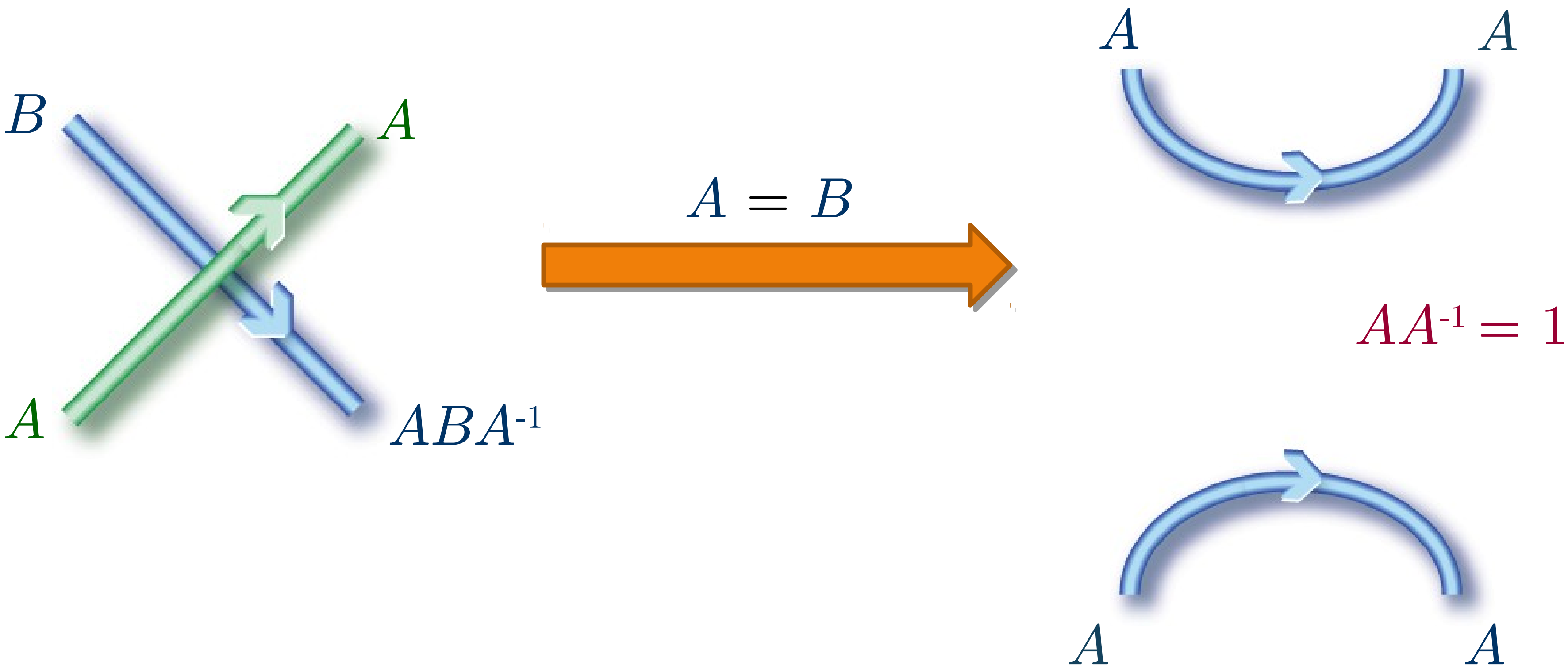
Passing dynamics is possible for Abelian case

Collision of Vortex



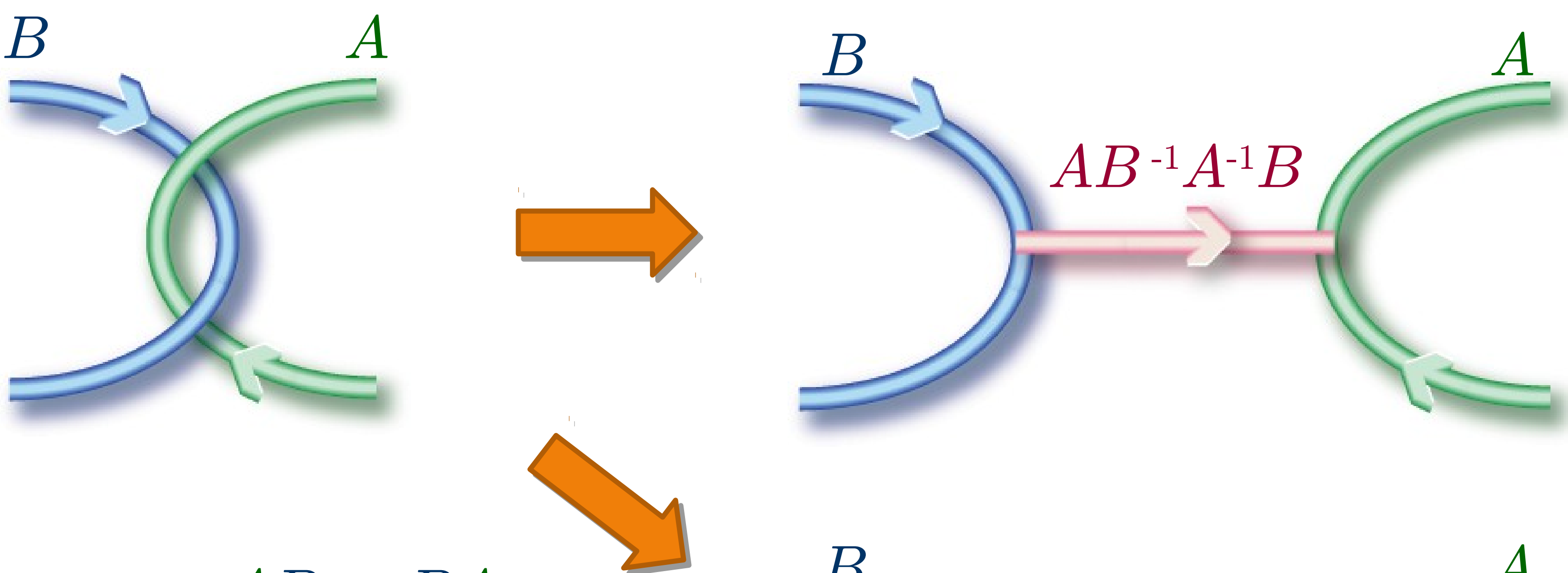
Rung BA^{-1} is formed through the collision.

Collision of Vortex



Rung disappears for the same charge resulting reconnection.

Linked Vortex Rings

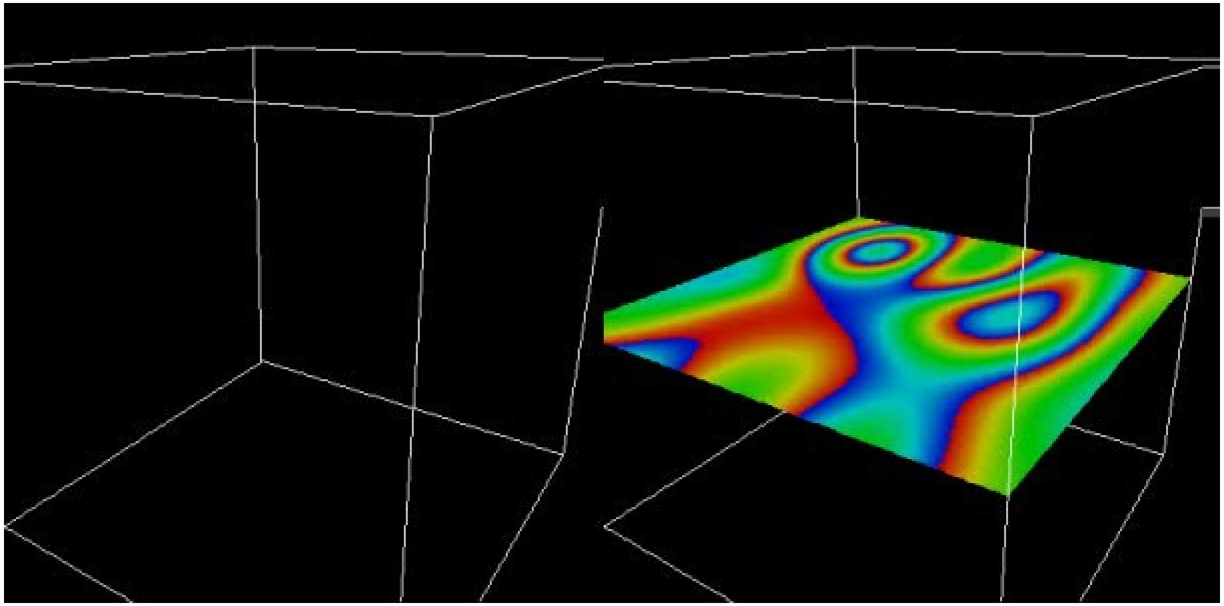


$$AB = BA$$

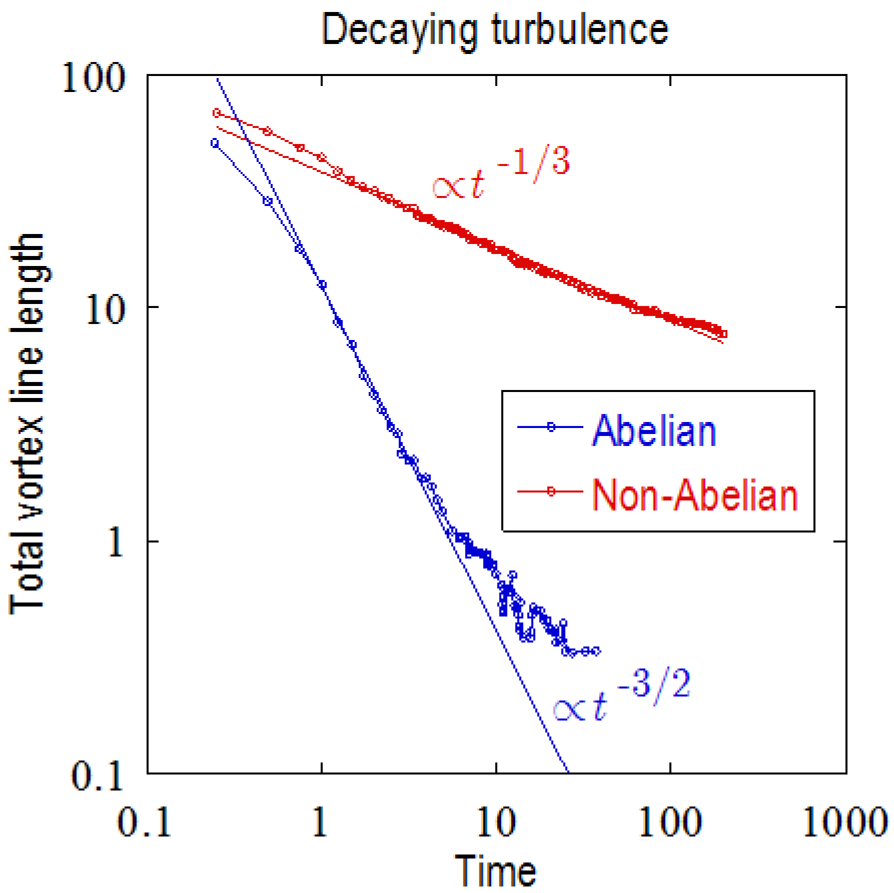
Linked excitations with non-Abelian invariants never unravel.

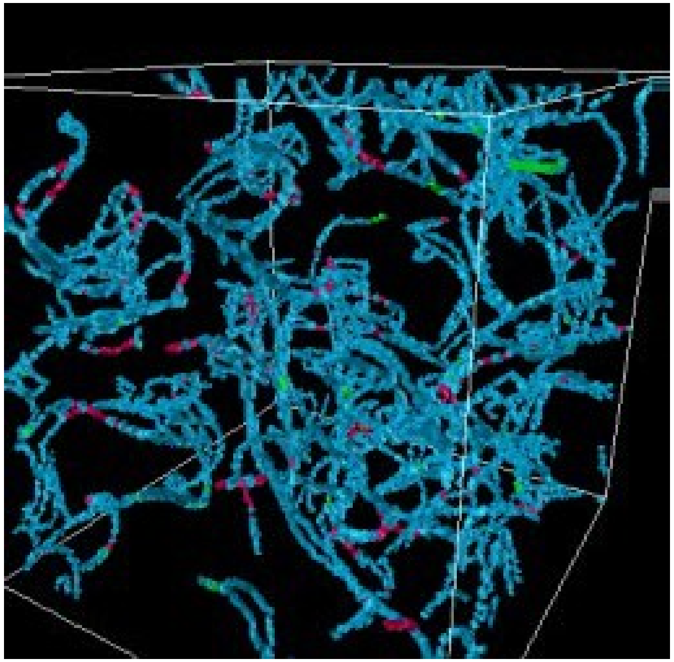
Quantum turbulence

Vortices 2D plane of the phase



Non-Abelian quantum turbulence





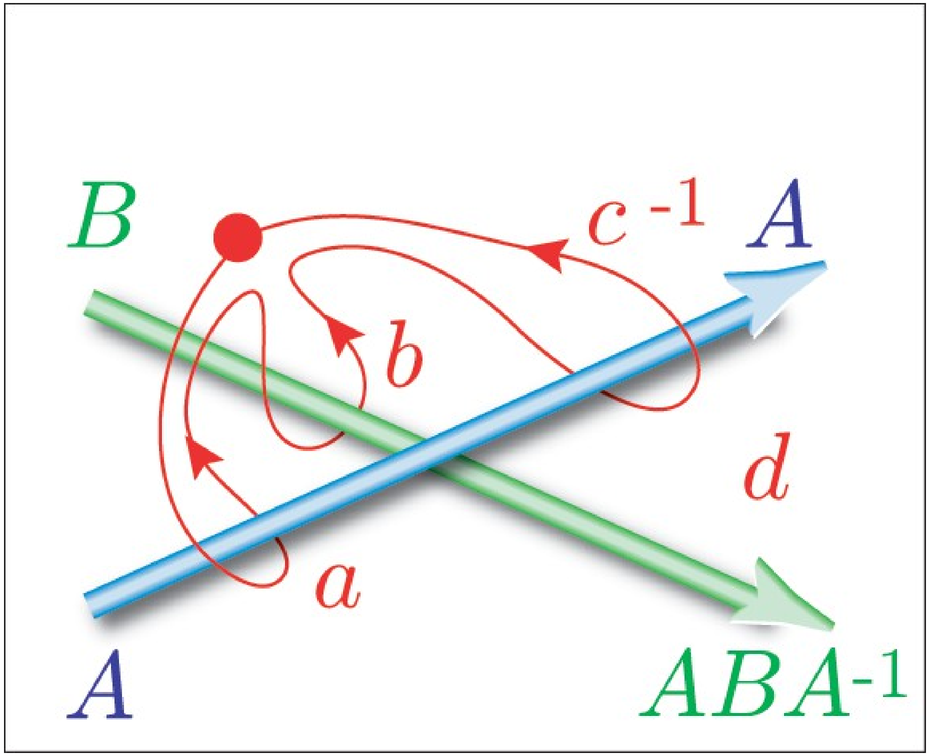
Turbulent behavior is strongly affected by topology

M. Kobayashi, et al. in press

Summary

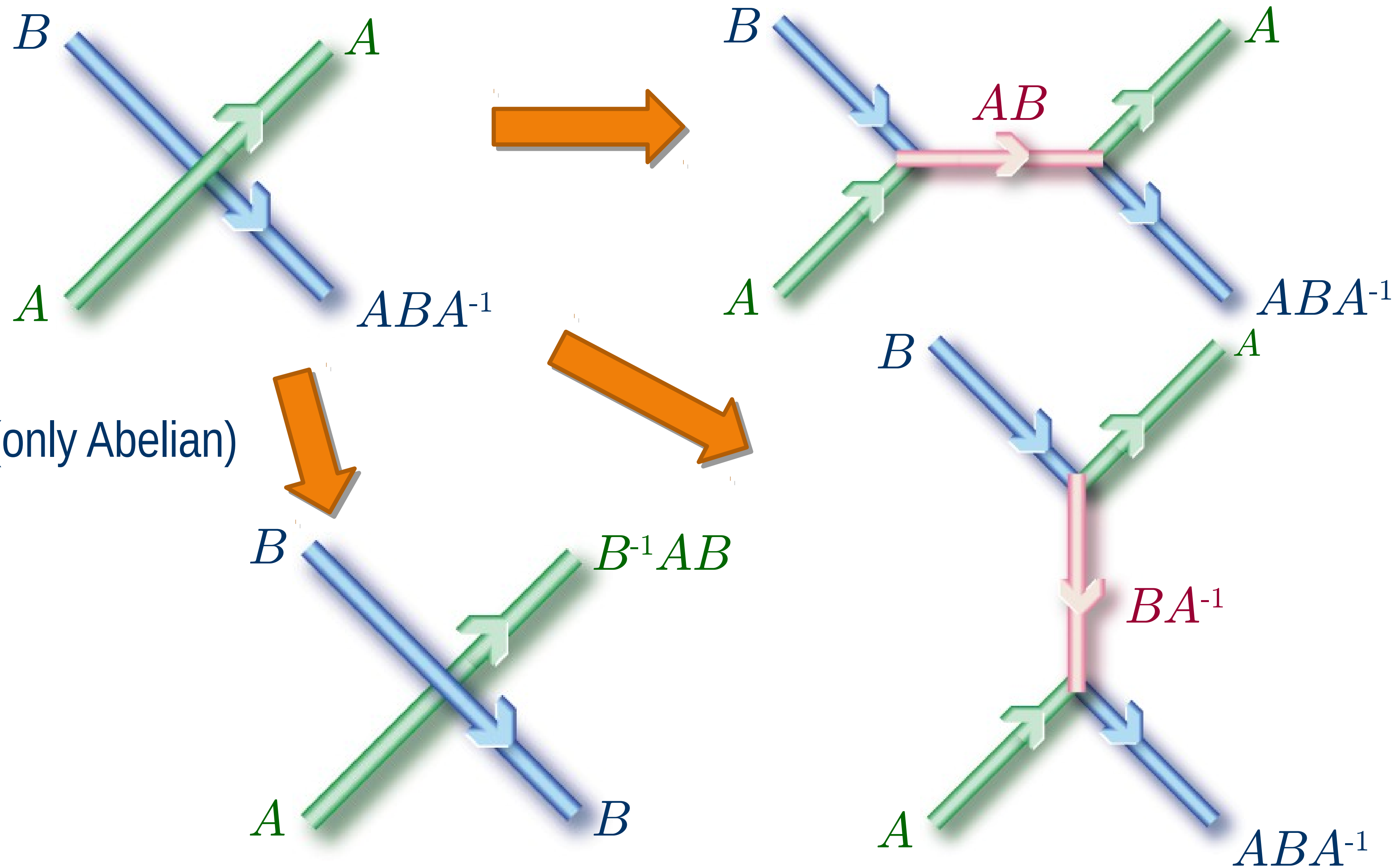
- In BEC, various kinds of topological excitations can be realized.
- Dynamics of topological excitations are affected by the order-parameter manifold and can dominate the nature of the system.

Topological charge of topological excitation



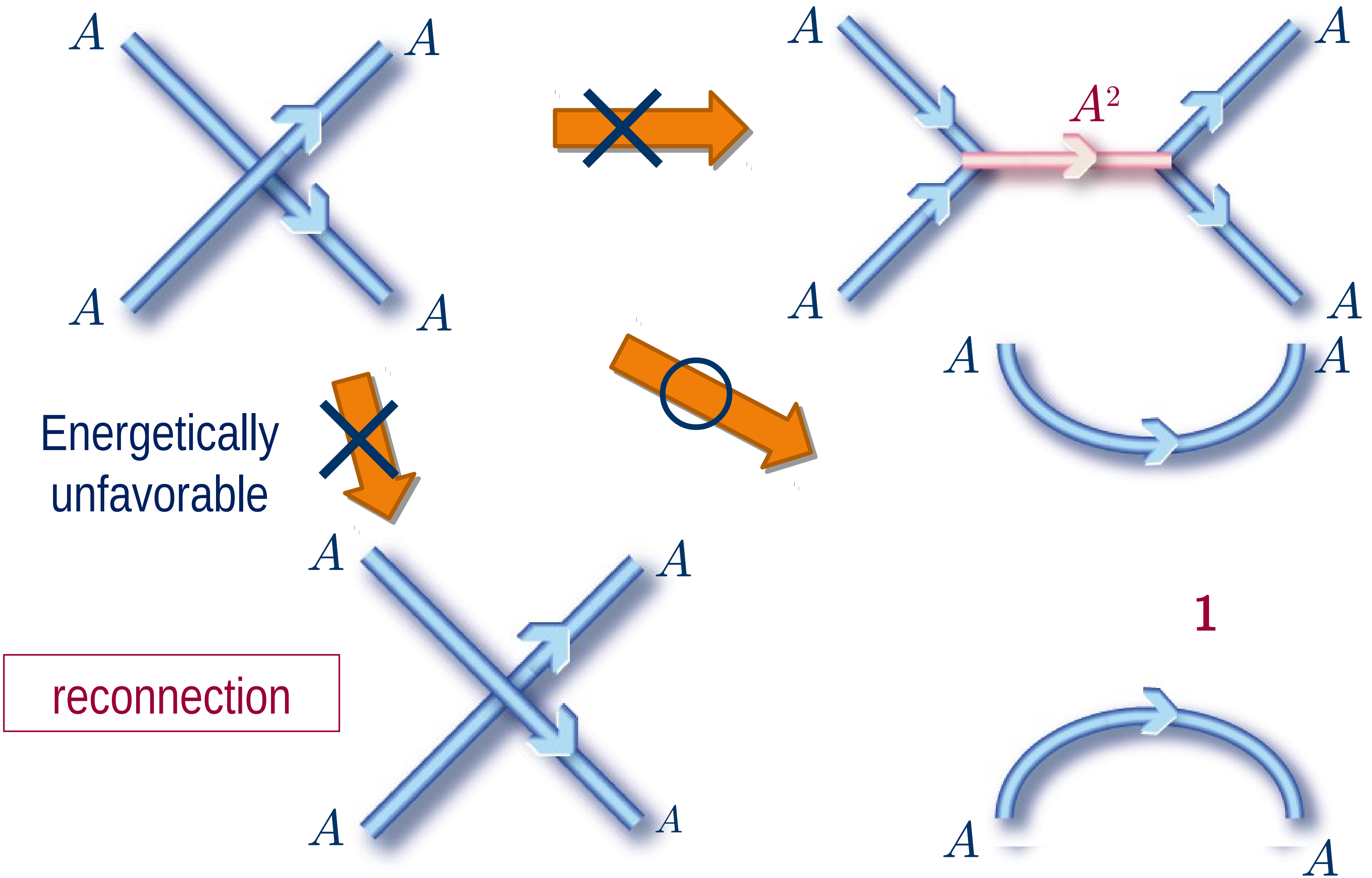
Topological charge of vortex can be fixed by a closed path encircling the vortex

Collision of Vortices

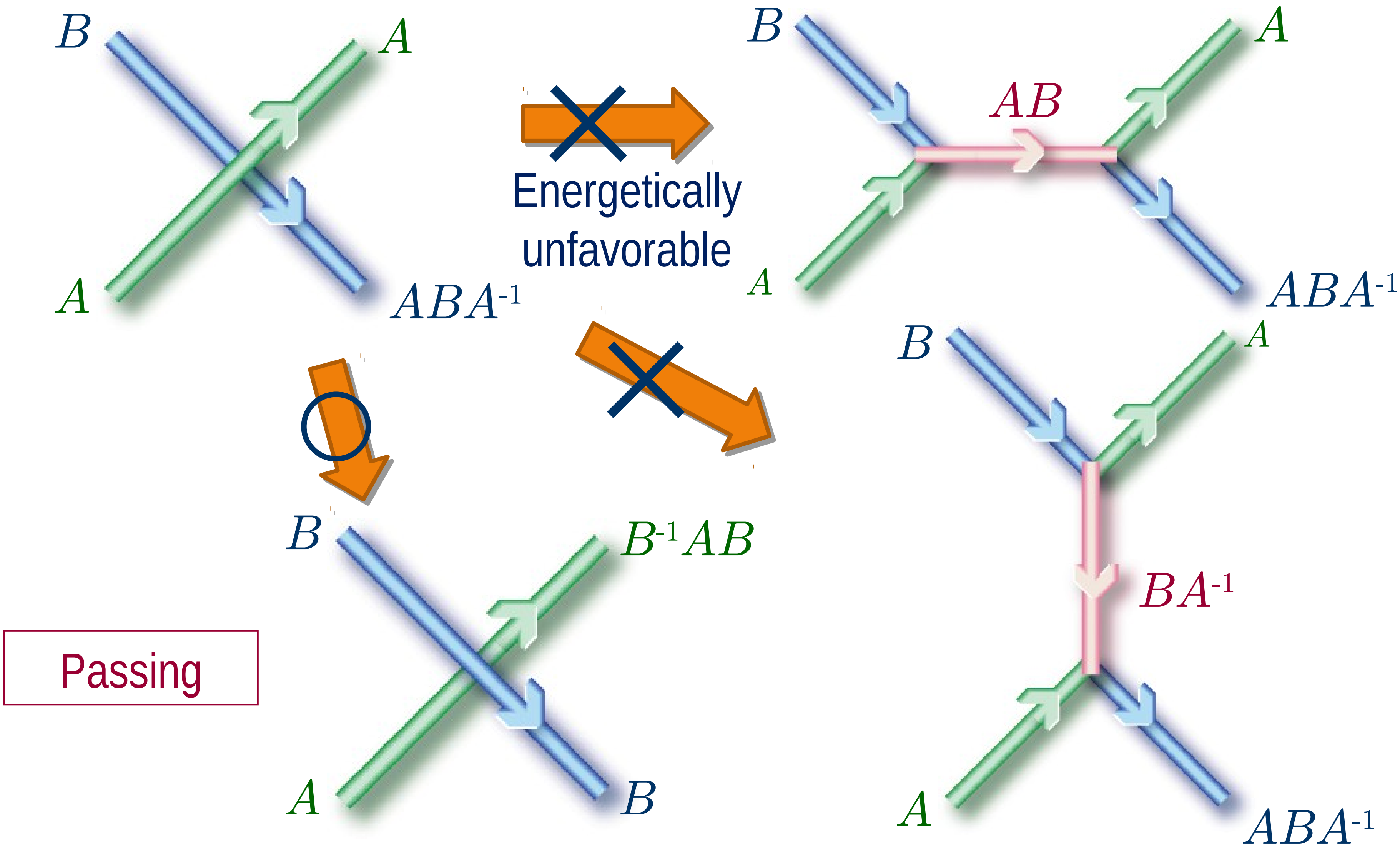


(only Abelian)

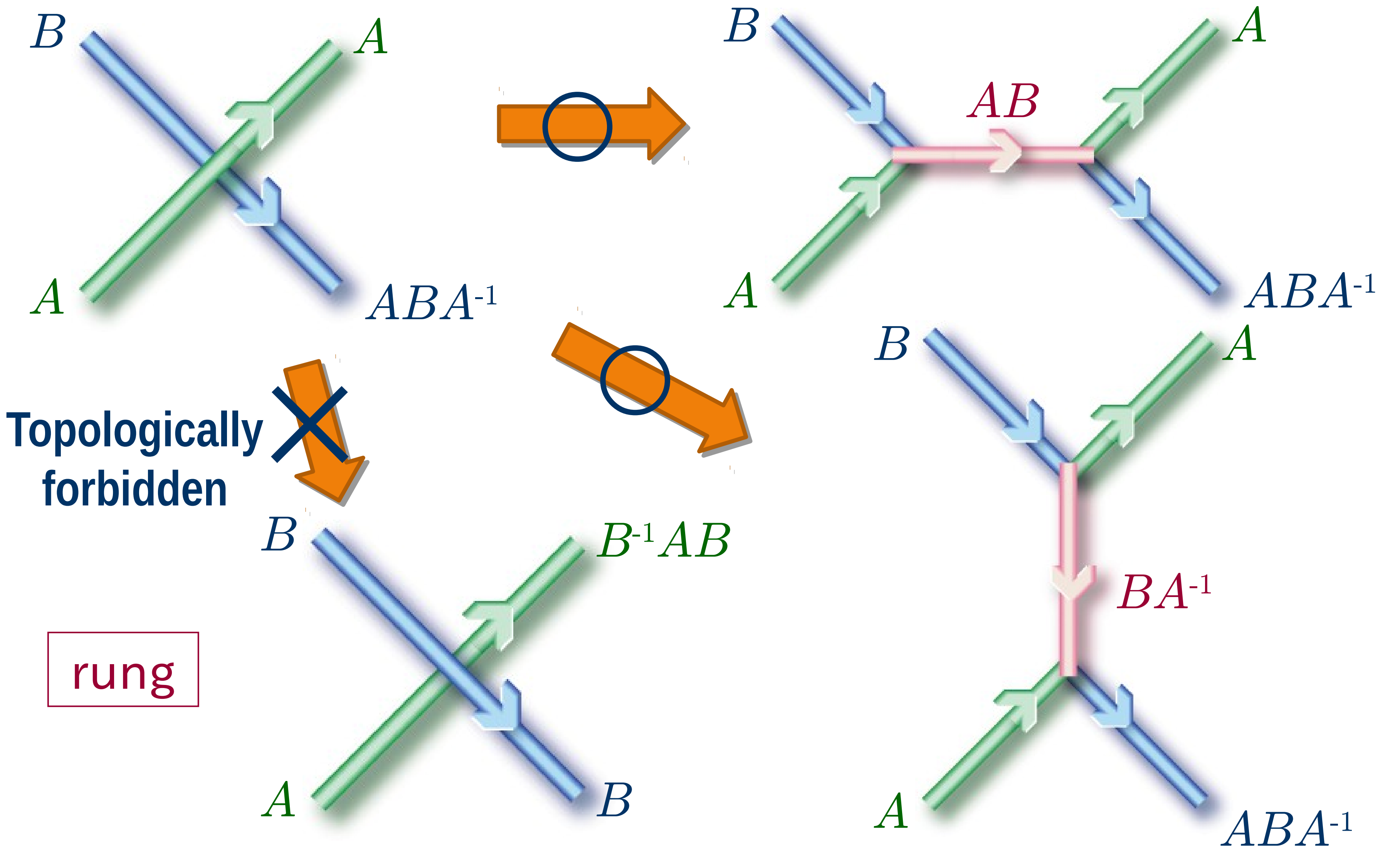
Collision of Same Vortices



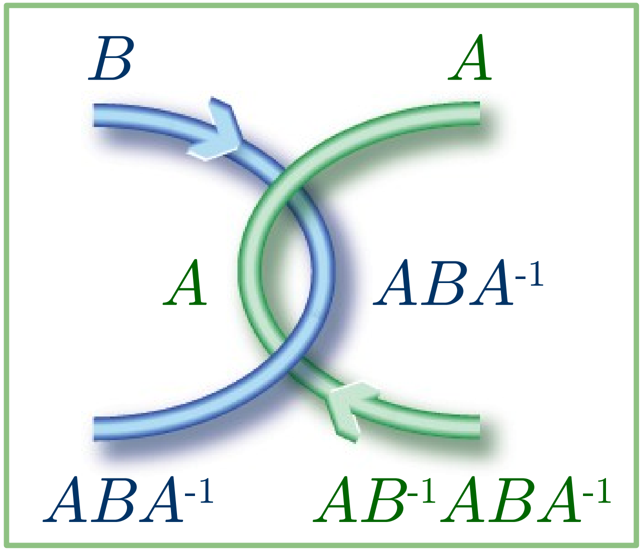
Collision of Different Commutative Vortices

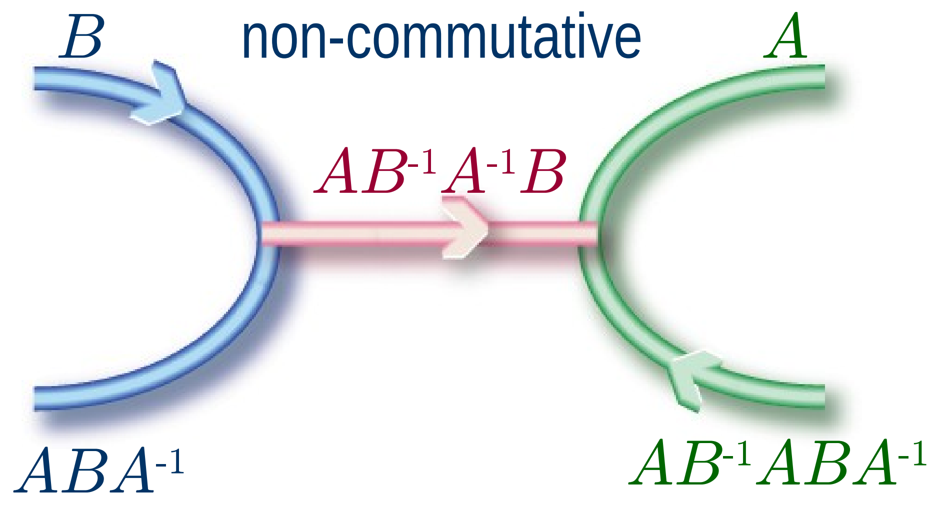


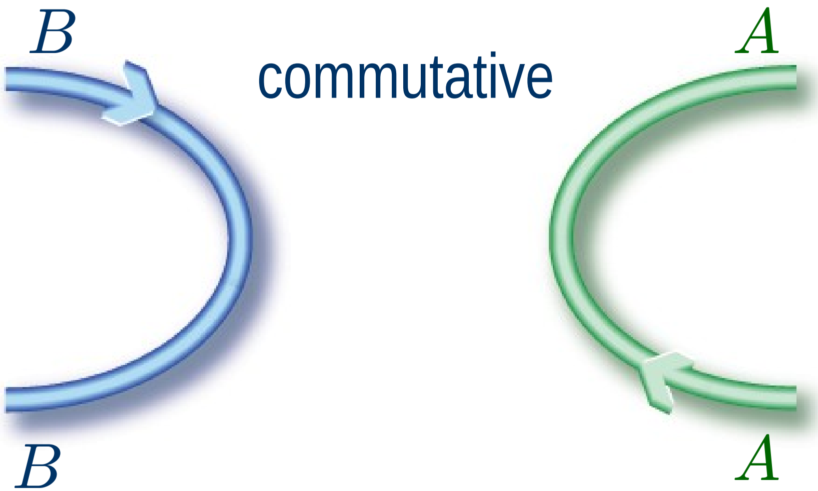
Collision of Different Non-commutative Vortices



Linked Vortices



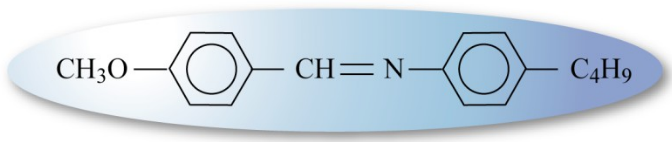


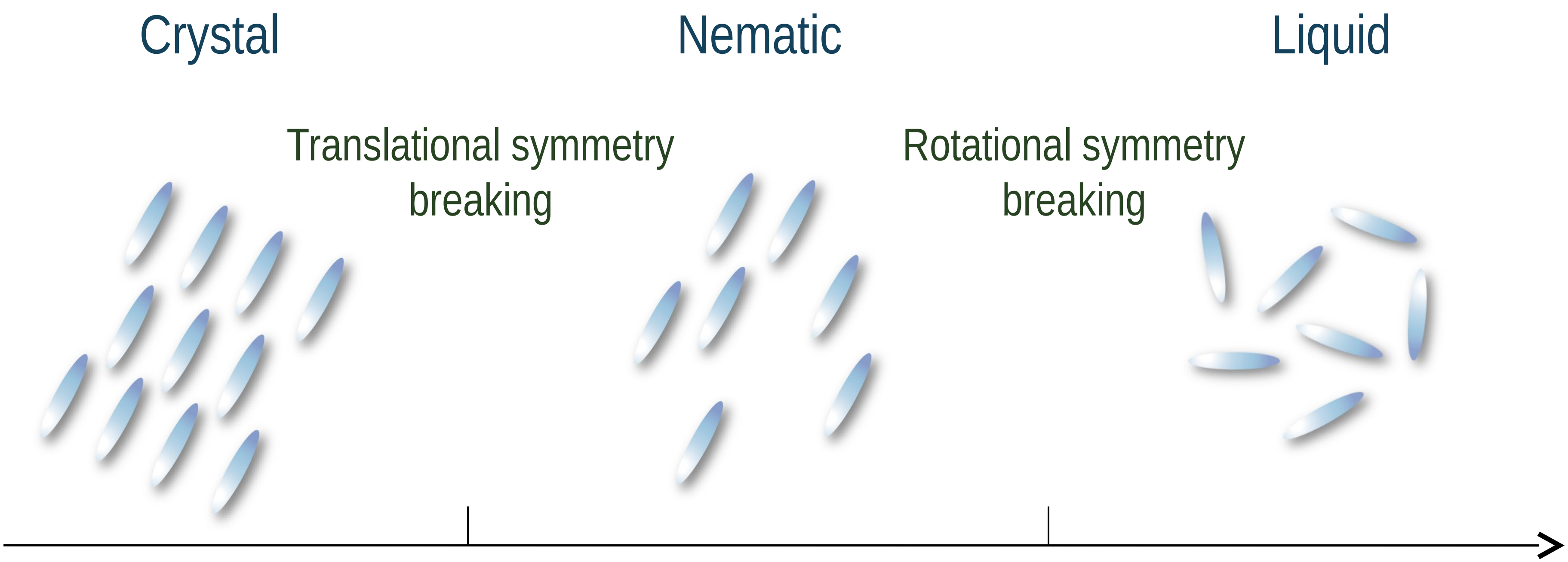


Linked vortices cannot untangle

What is topological excitations?

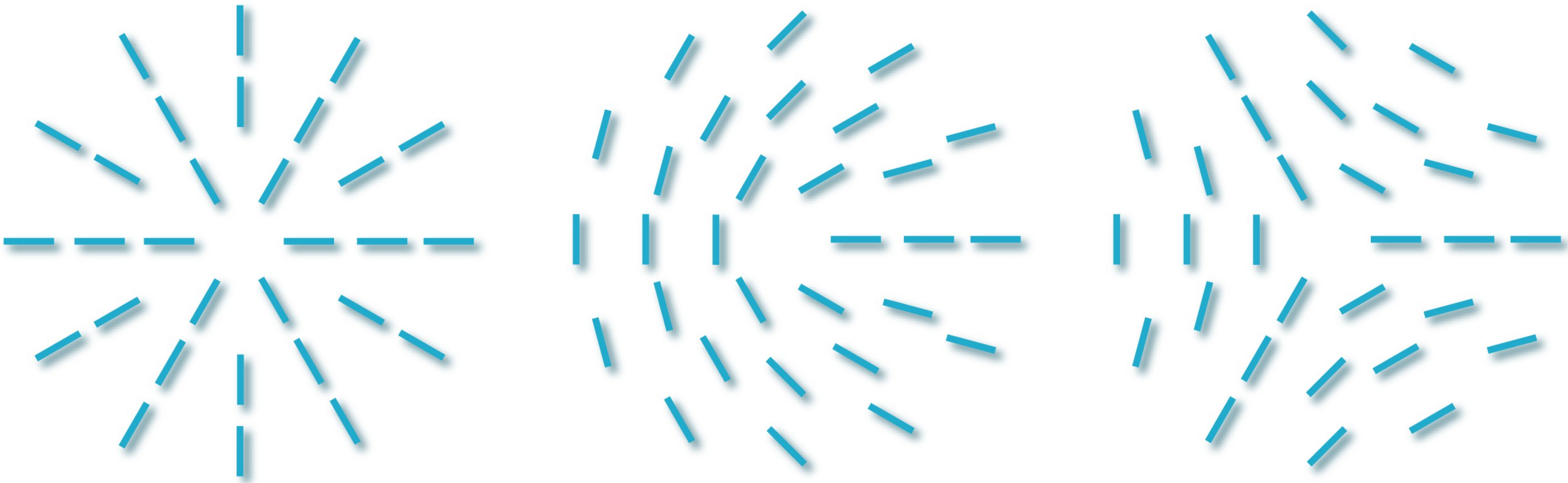
Nematic liquid crystal





What is topological excitations?

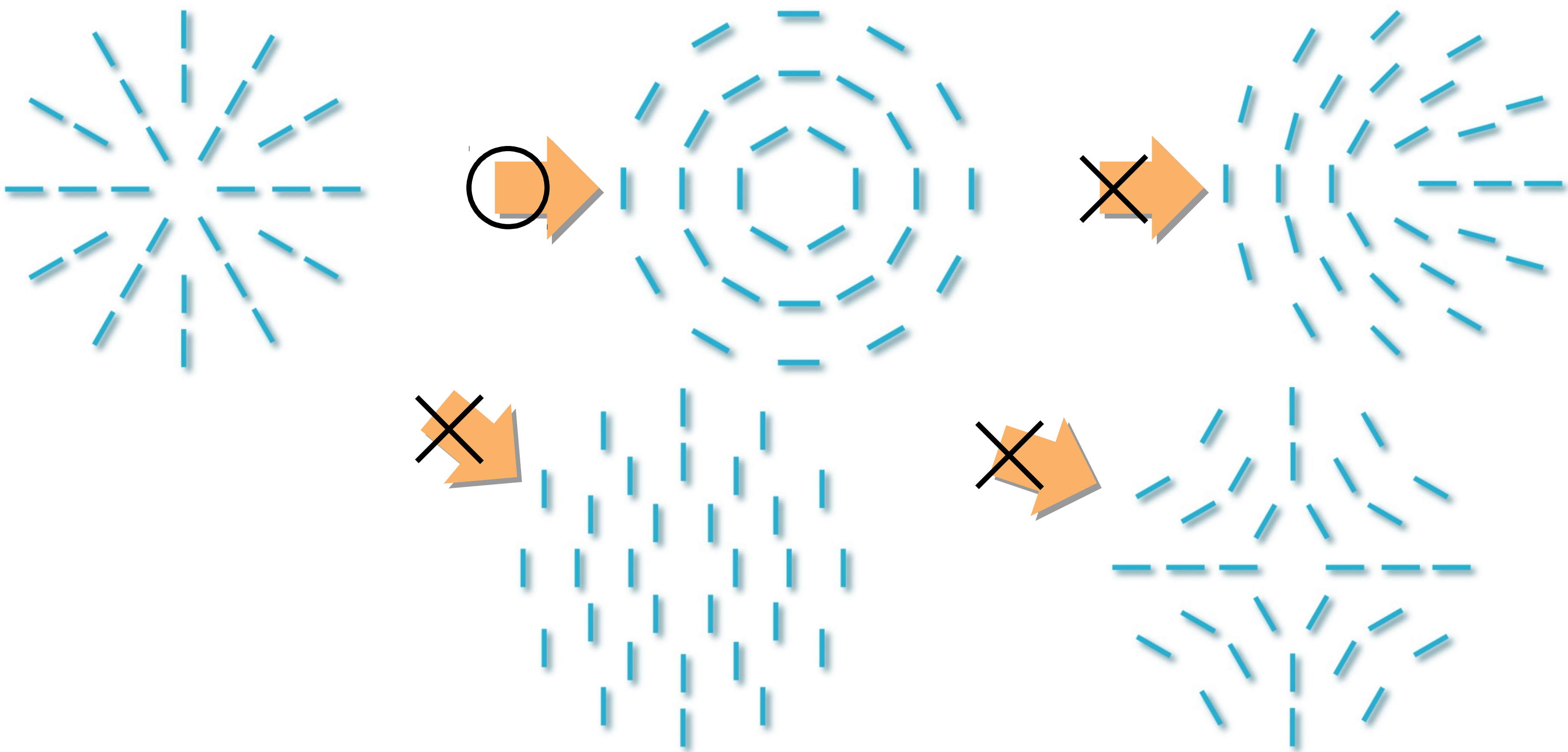
Topological excitations in nematic liquid crystal



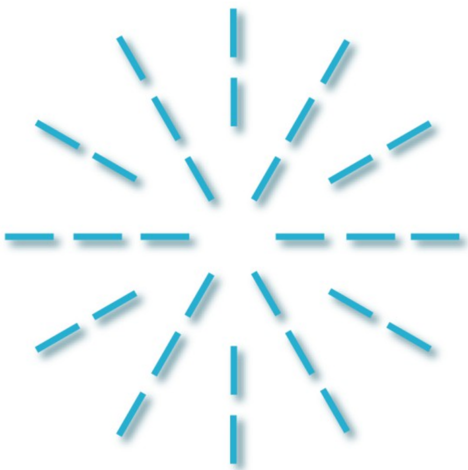
Topological excitation related to rotational symmetry breaking

What is topological excitations?

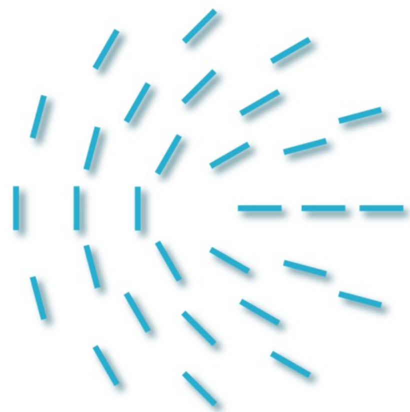
States with topological excitations cannot be continuously transformed to states without topological excitations



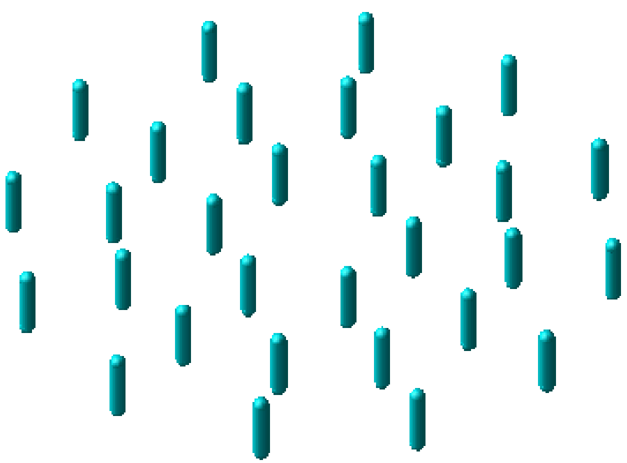
What is topological excitations?



This is topological excitation in 2D system but not topological excitation in 3D system

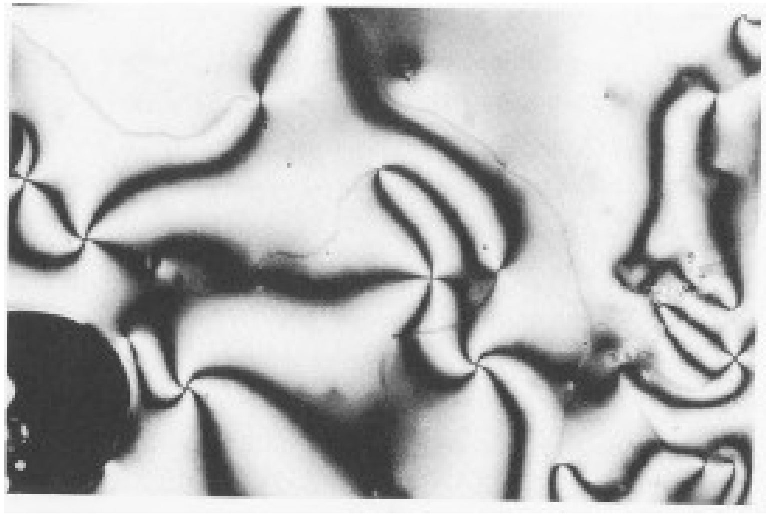


This is always topological excitation

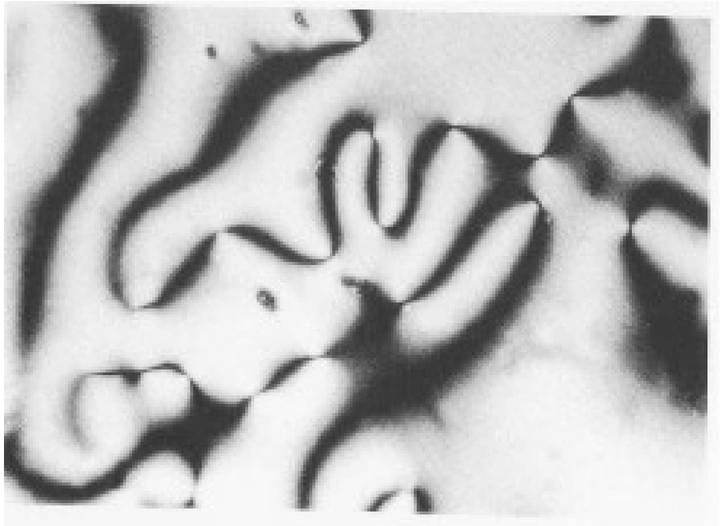


Characteristics of topological excitations strongly depend on the internal degrees of freedom (topology) of the system

Observation of topological excitations in nematic liquid crystal

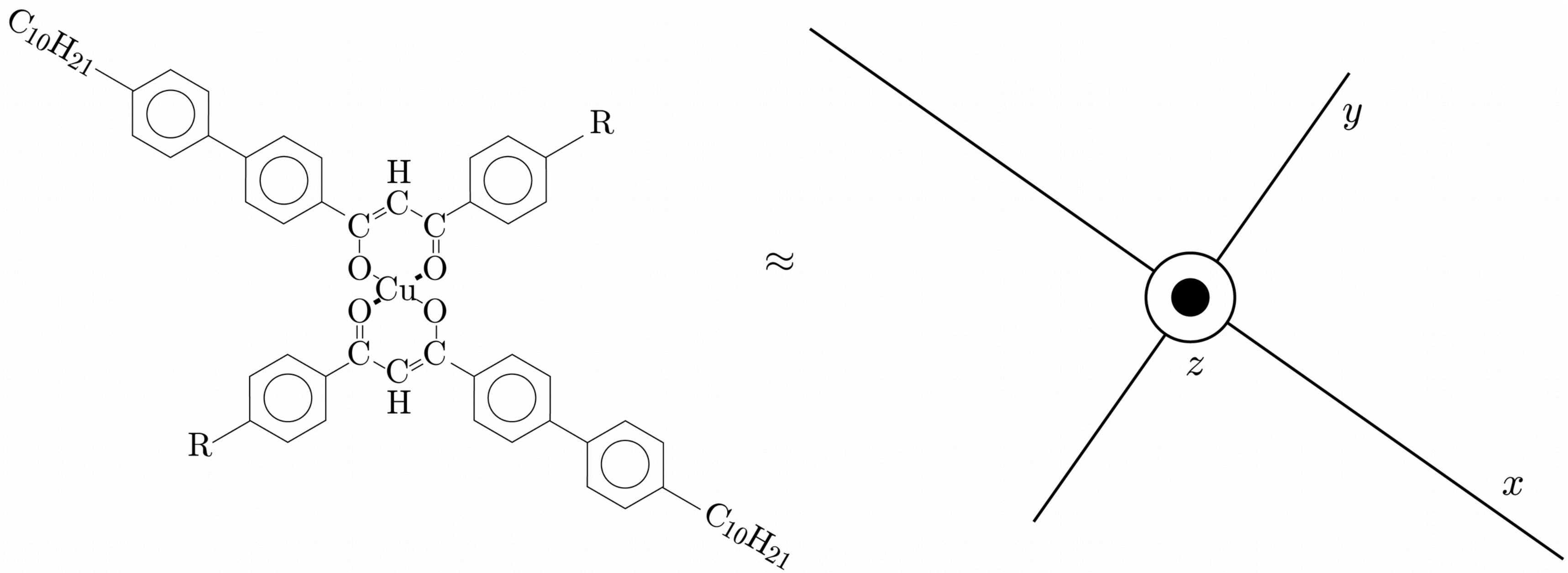


near surface



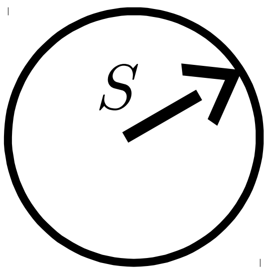
far from surface

Biaxial nematic liquid crystal



Topological excitations and homotopy

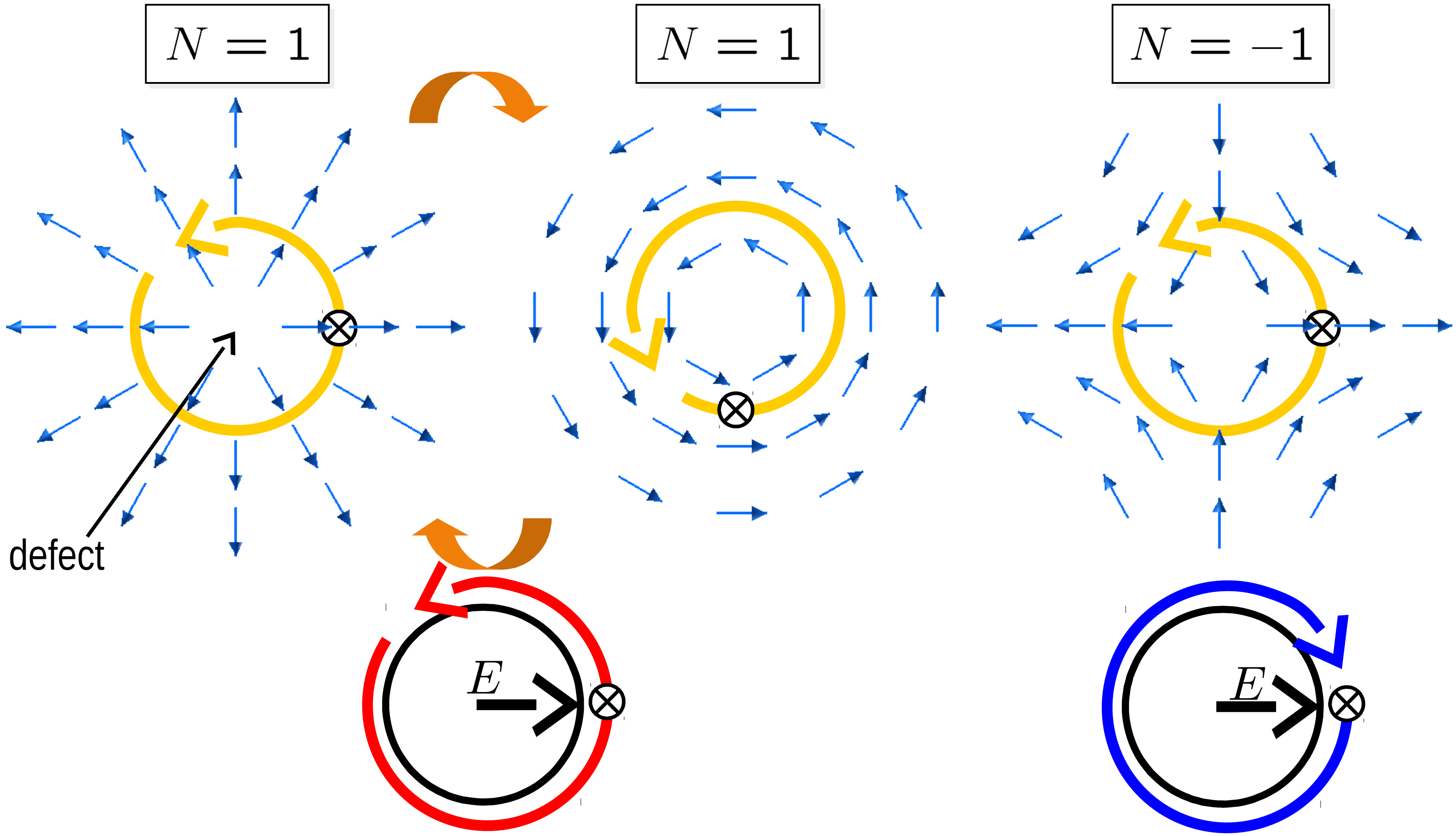
XY-spin system



In XY-spin system, local spin (order parameter) can be expressed by a point in a circle

$\longrightarrow$Order-parameter manifold

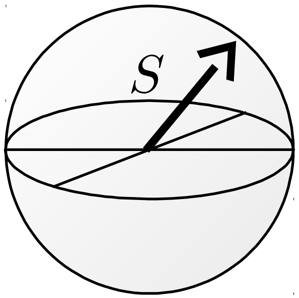
Topological defects and homotopy

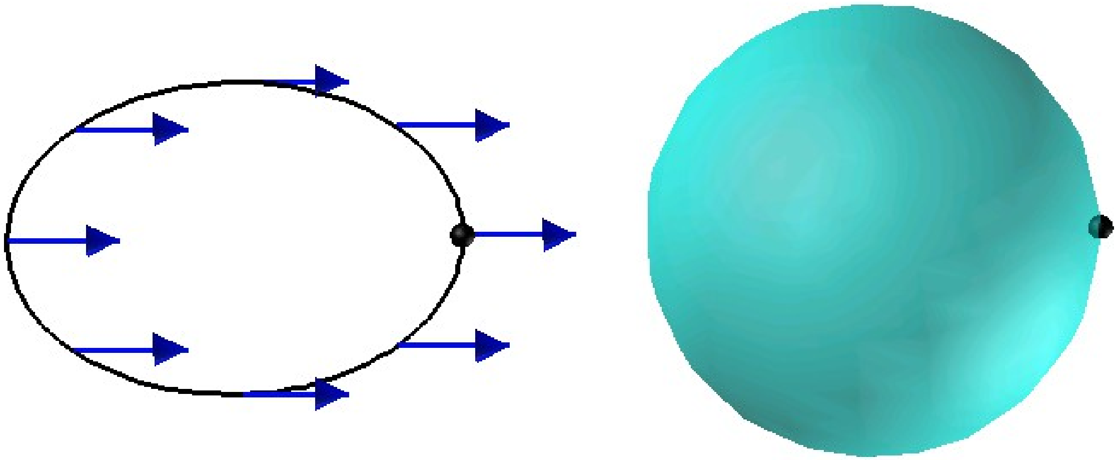


Topological excitations can be characterized by how many times the state rotates the circle along the closed path

Heisenberg-spin

Order parameter can be expressed by a point in a sphere





Topological excitations can never be stabilized

What is topological excitations?

Excitations in symmetry broken systems via phase transitions

Liquid → Solid transition (spontaneous symmetry breaking)

Liquid

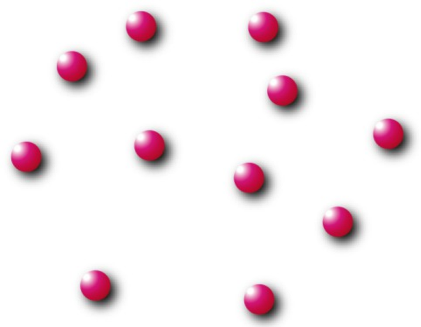
Solid (crystal)

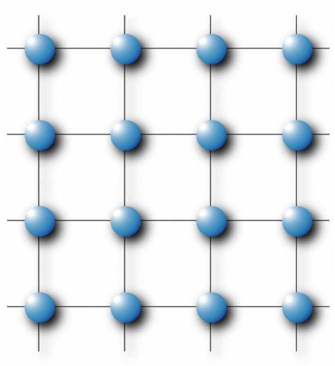
- Free energy is invariant under translational and rotational transformations
- System is also invariant under transformations

- Free energy is invariant under translational and rotational transformations
- System is not invariant under transformations (symmetry breaking)

What is topological excitations?

Liquid





Solid

Atoms are little influenced by other atoms.

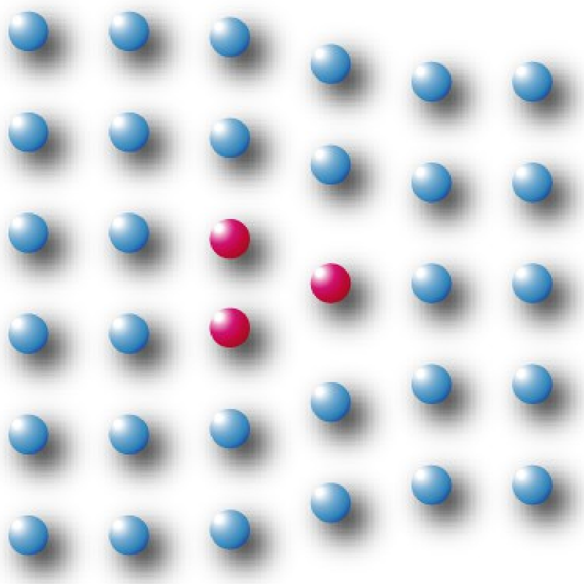
Positions and orientations of atoms are strongly affected by other atoms and fixed (spontaneous symmetry breaking).

What is topological excitations?

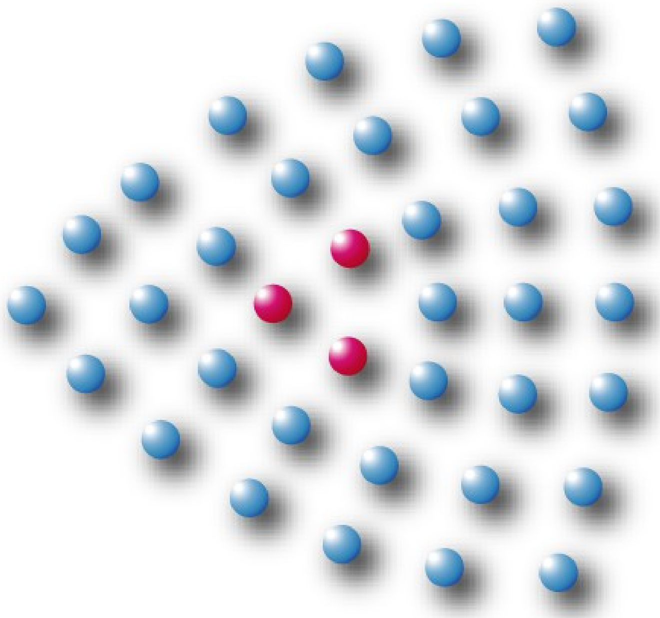
Topological excitations appear in symmetry broken systems

In crystal

dislocation	disclination



Topological excitation related to translational symmetry breaking



Topological excitation related to rotational symmetry breaking

$U(1)$ gauge symmetry breaking in BEC

Mean-field Hamiltonian at the zero temperature

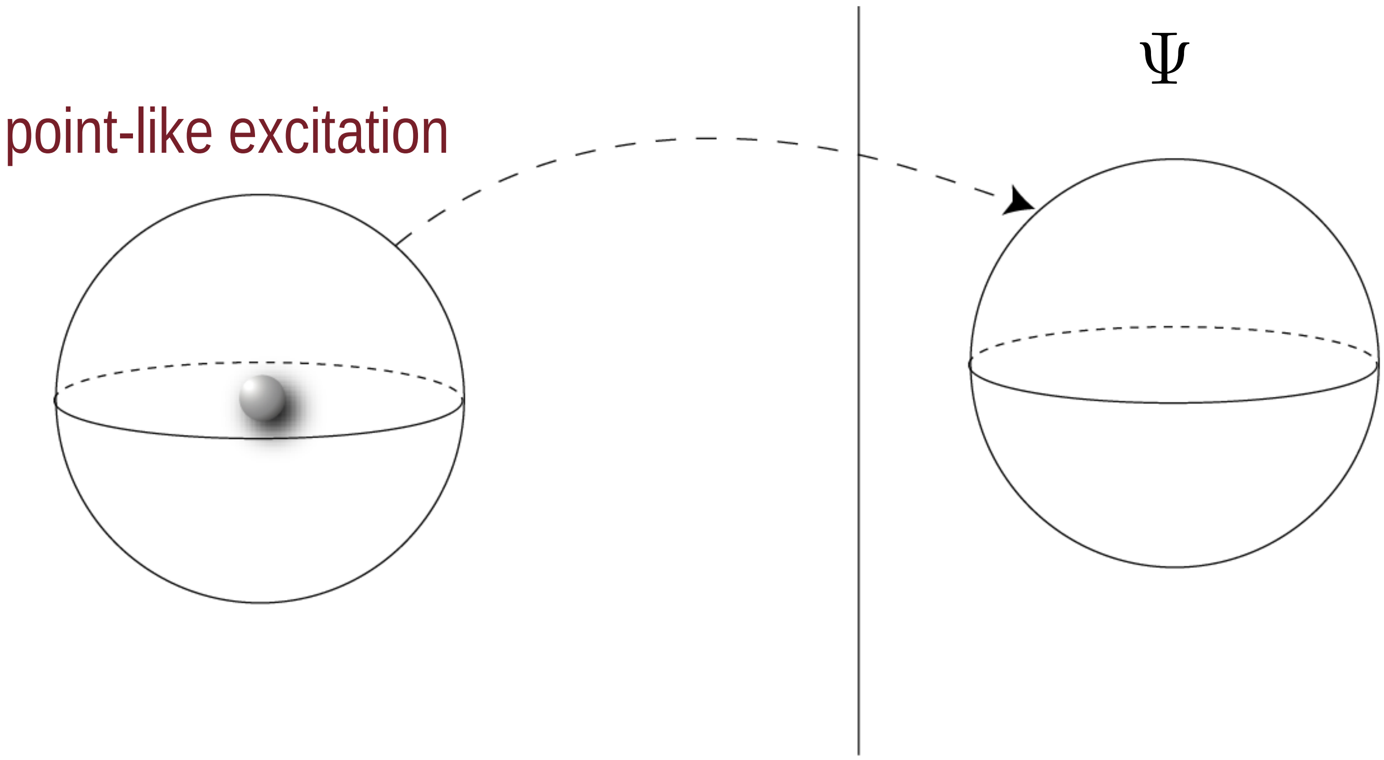
$$H = \int d\boldsymbol{x} \left[\frac{\hbar^2}{2M} \nabla \Psi^*(\boldsymbol{x}) \nabla \Psi(\boldsymbol{x}) + \frac{c_0}{2} |\Psi(\boldsymbol{x})|^4 \right]$$

$$\Psi(\boldsymbol{x}) = |\Psi(\boldsymbol{x})| \exp[i\varphi(\boldsymbol{x})]$$

$$\rho(\boldsymbol{x}) = |\Psi(\boldsymbol{x})|^2 \text{ : Fluid density}$$

$$\boldsymbol{v}(\boldsymbol{x}) = \frac{\hbar}{m} \nabla \varphi(\boldsymbol{x}) \text{ : Fluid velocity}$$

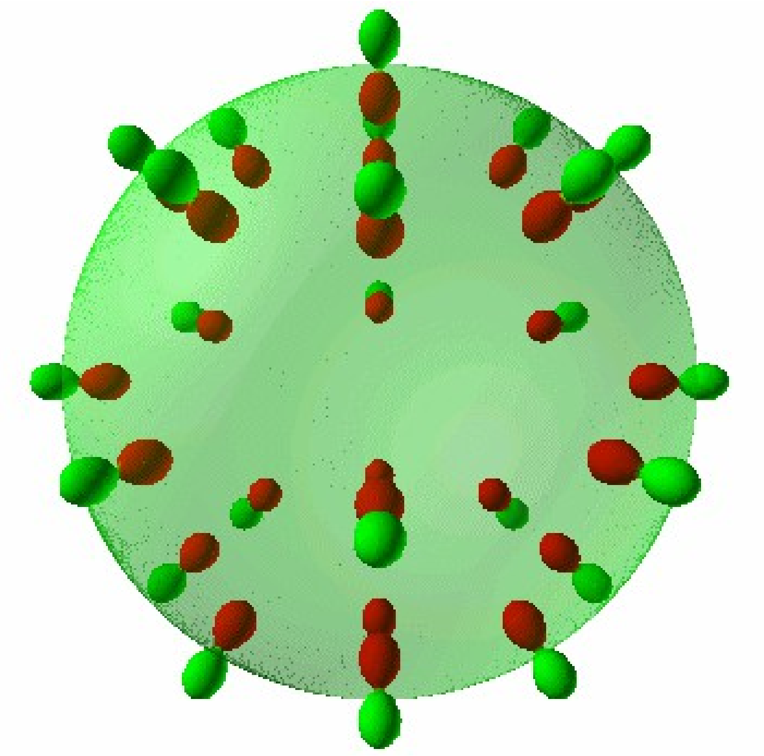
Point-like excitation



Point-like excitation

Polar phase

$$\frac{G}{H} \simeq \frac{U(1)_\varphi \times S^2_F}{(\mathbb{Z}_2)_{\varphi+F}}$$

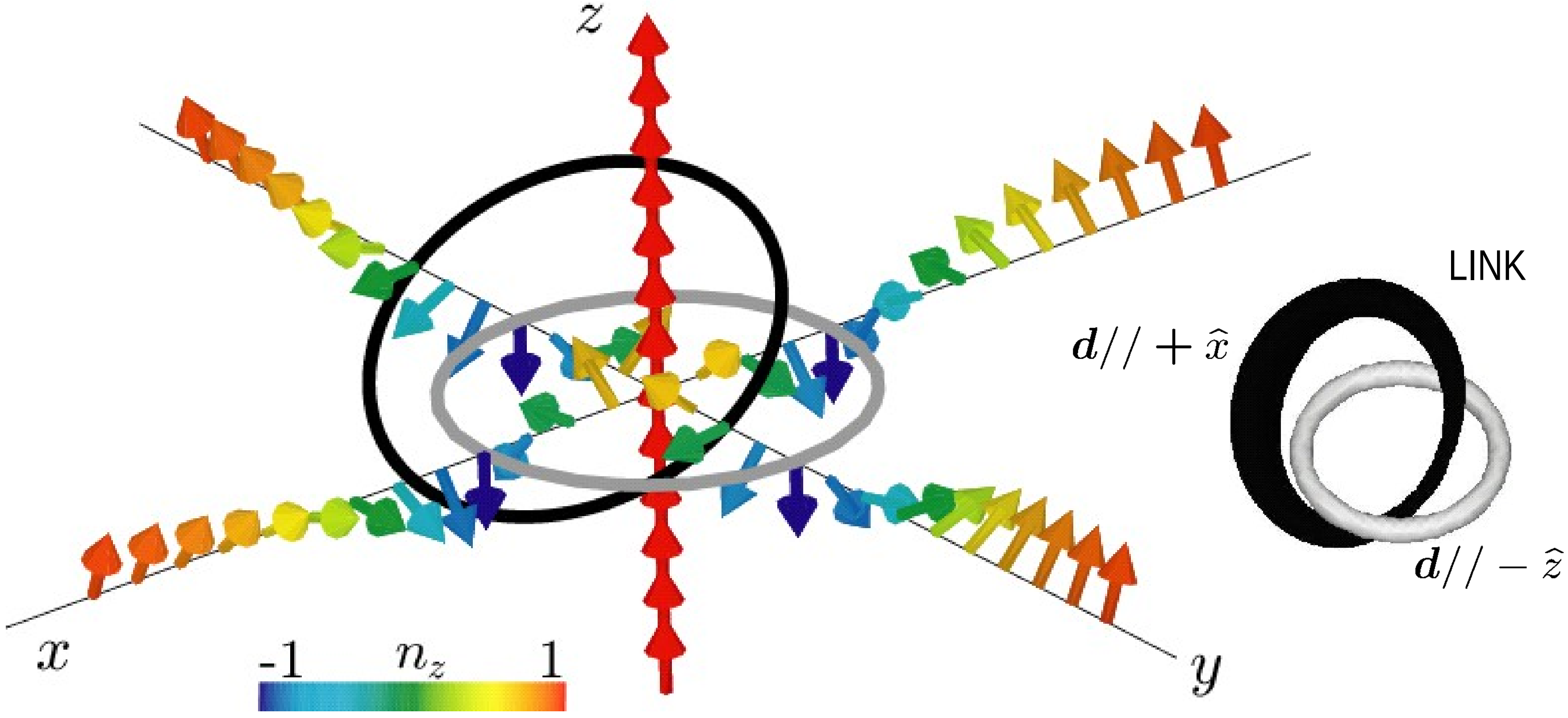


Point-like excitation cannot exist in Ferromagnetic phase

$$\frac{G}{H} \simeq SO(3)_{\varphi+F}$$

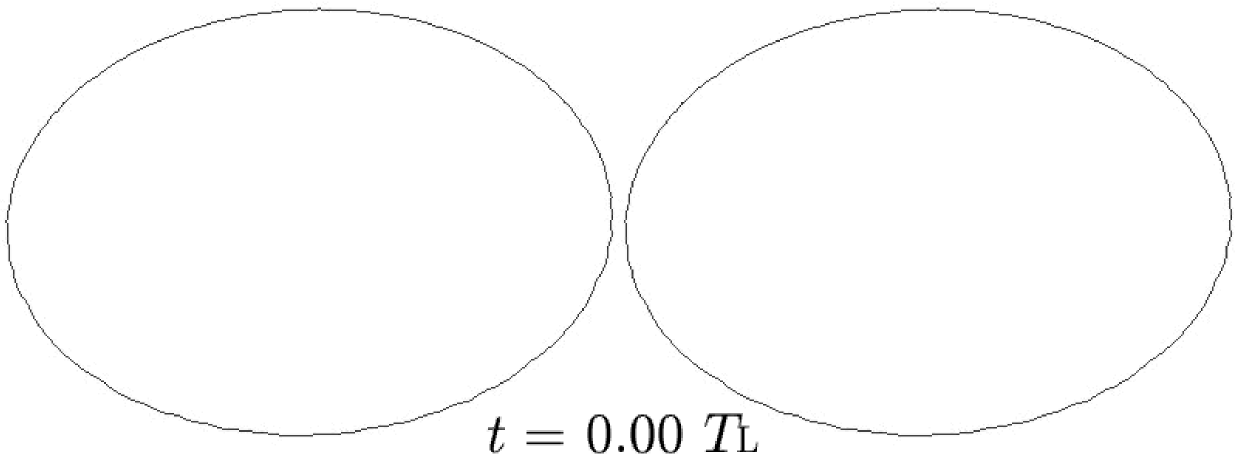
π_3 excitation and Hopf mapping

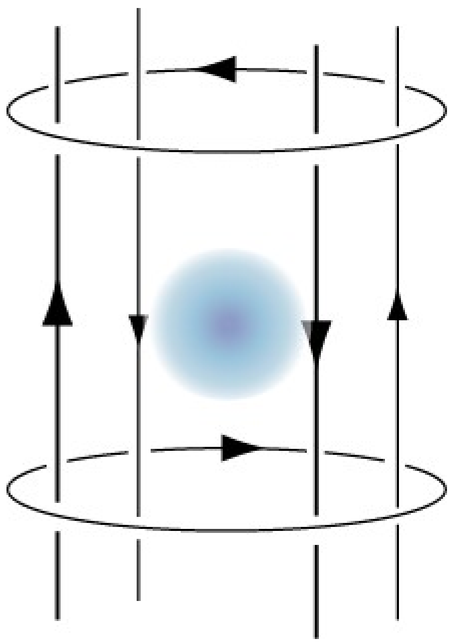
Y. Kawaguchi, et al., PRL **100**, 180403 (2008)

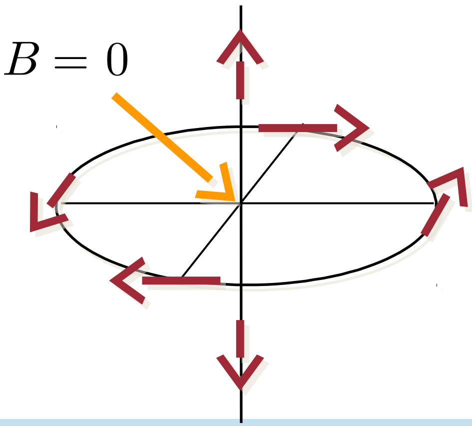


$$\pi_3\left[\frac{U(1)_{\mathrm{G}}\times(S^2)_{\mathrm{S}}}{(\mathbb{Z}_2)_{\mathrm{G+S}}}\right]\cong(\mathbb{Z})_{\mathrm{S}}$$

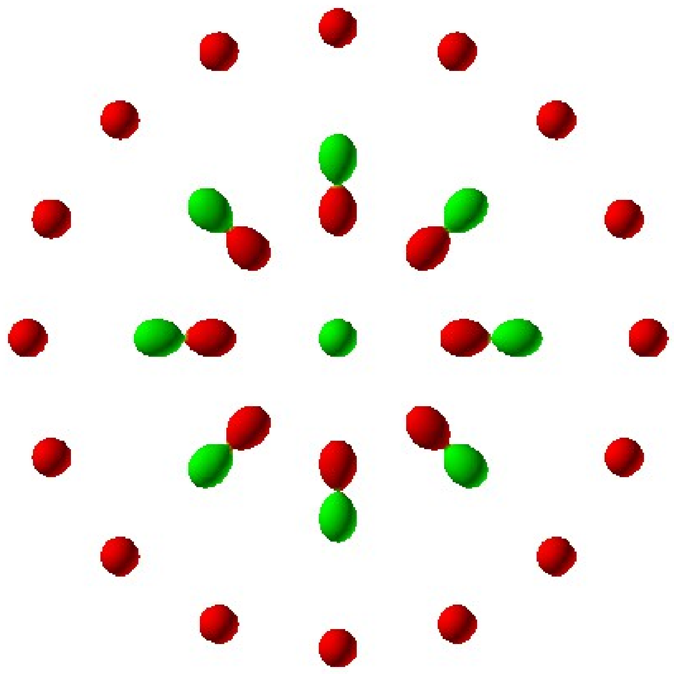
π_3 excitation and Hopf mapping

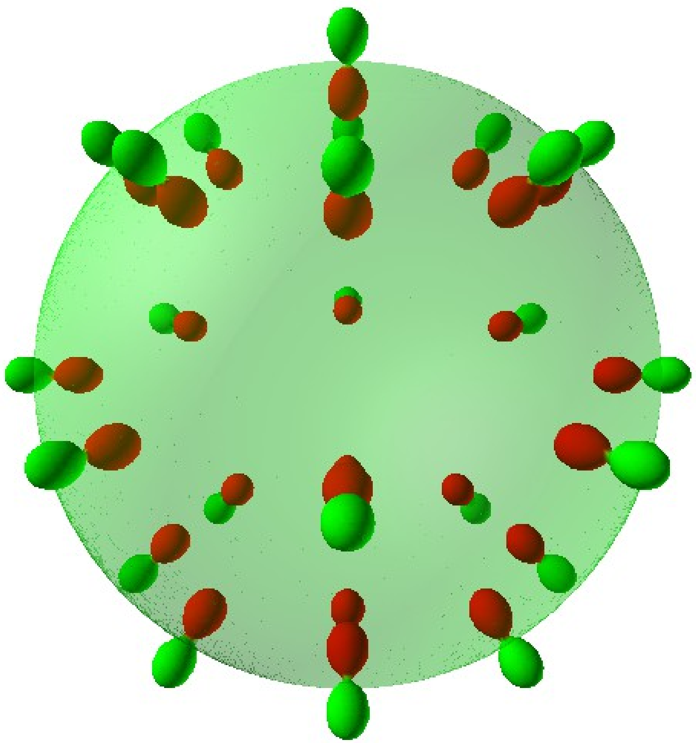






Point-like excitation and 2D skyrmion

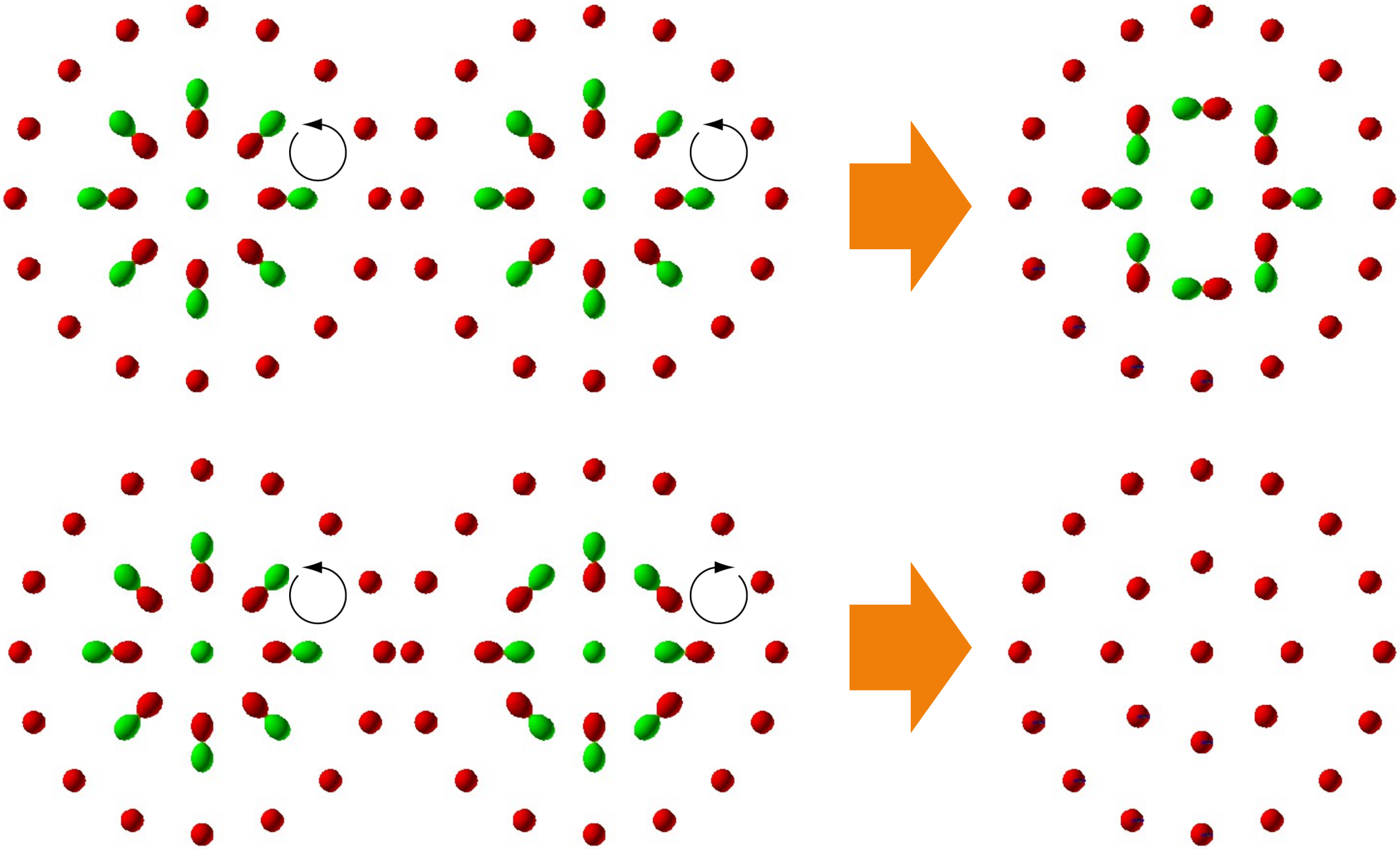


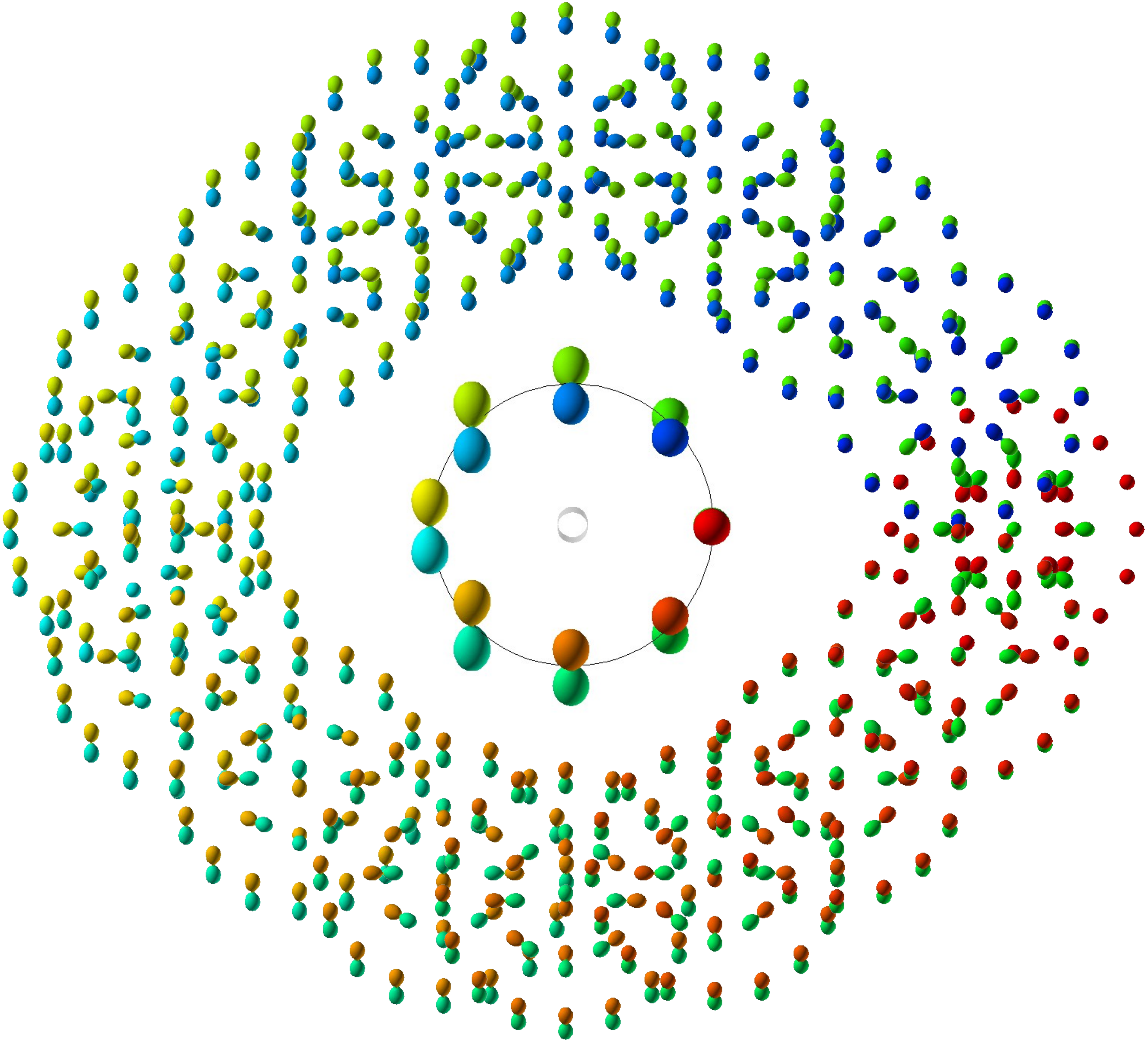


L. S. Leslie, et al. arXiv:0910.4918

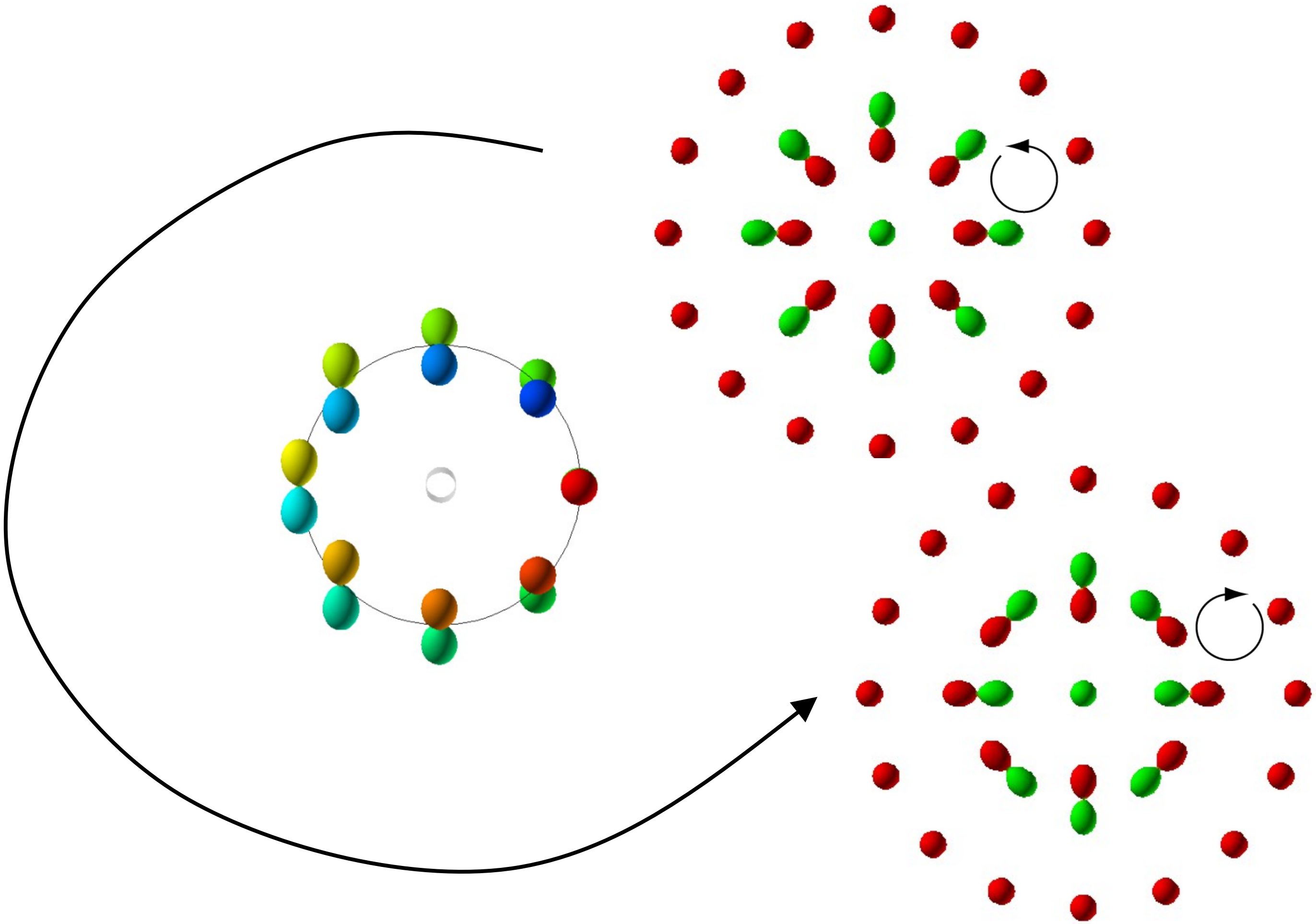
$$\pi_2 \left[\frac{U(1)_{\mathrm{G}} \times (S^2)_{\mathrm{S}}}{(\mathbb{Z}_2)_{\mathrm{G+S}}} \right] \cong \pi_2[(S^2)_{\mathrm{S}}] \cong (\mathbb{Z})_{\mathrm{S}}$$

Two 2D skyrmion



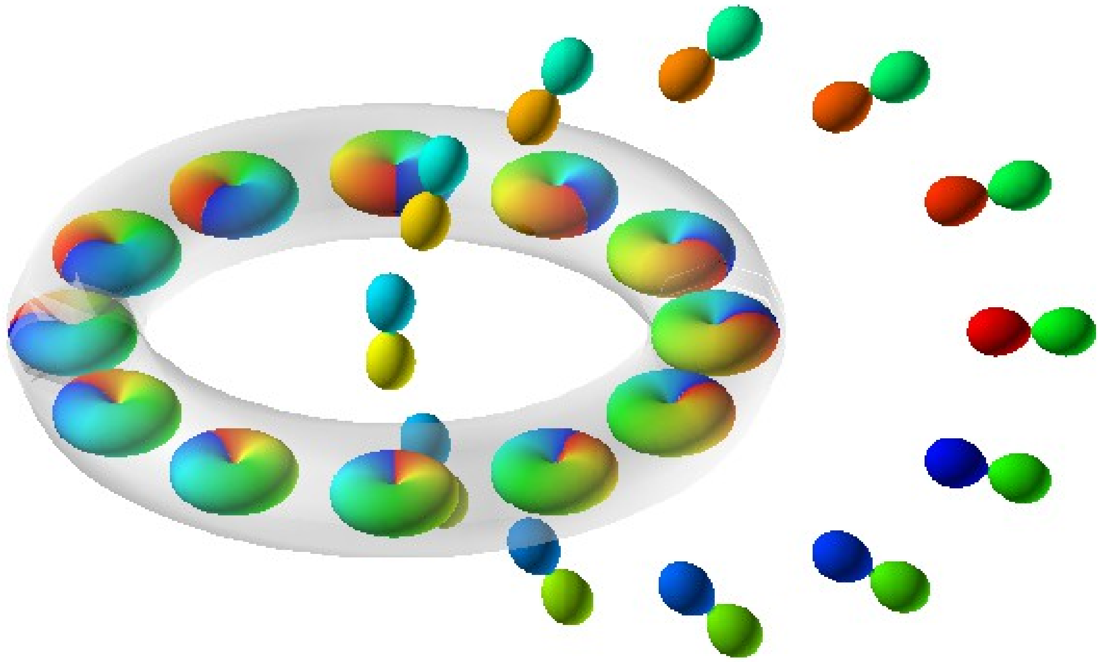


Inversion of topological invariant



Vorton excitation

vorton

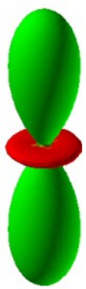


Spin-2 case

$$H = \int d\boldsymbol{x} \left[-\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + \frac{c_0}{2}\rho^2 + \frac{c_1}{2}\boldsymbol{F}^2 + \frac{c_2}{2}|A_{20}|^2 \right]$$

c_1

Uniaxial Nematic:

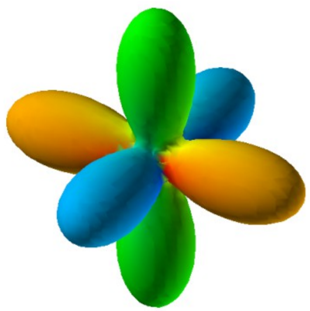


$$\Psi_{\rm U} = (0,0,1,0,0)^T$$

$$U(1)_\varphi \times \frac{S_F^2}{(\mathbb{Z}_2)_F}$$

Cyclic:

$$\Psi_{\rm C} = (1,0,0,\sqrt{2},1)^T/\sqrt{3}$$



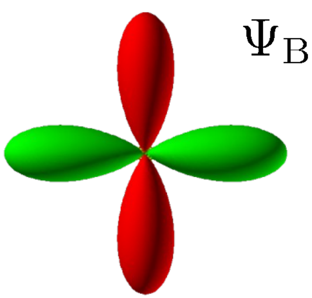
$$\frac{U(1)_\varphi \times SO(3)_F}{(T)_{\varphi+F}}$$

^{87}Rb

c_2

Biaxial Nematic:

$$\Psi_{\rm B} = (1,0,0,0,1)^T/\sqrt{2}$$



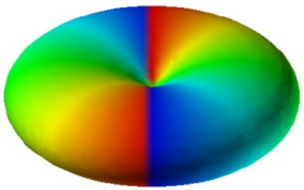
$$\frac{U(1)_\varphi \times SO(3)_F}{(D_4)_{\varphi+F}}$$

$$c_2 = 4c_1$$

Ferromagnetic:

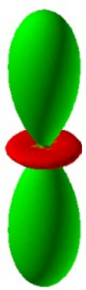
$$\Psi_{\rm F} = (1,0,0,0,0)^T$$

$$\frac{SO(3)_{\varphi+F}}{(\mathbb{Z}_2)_{\varphi+F}}$$



Nematic phase of spin-2

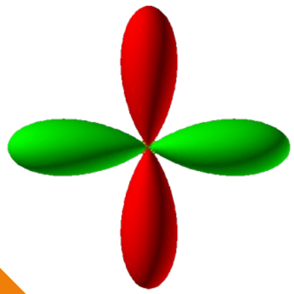
$$H = \int d\boldsymbol{x} \left[-\Psi_m^* \frac{\hbar^2}{2M} \nabla^2 \Psi_m + \frac{c_0}{2}\rho^2 + \frac{c_1}{2}\boldsymbol{F}^2 + \frac{c_2}{2}|A_{20}|^2 \right]$$



Uniaxial Nematic:

$$\Psi_{\mathrm{U}} = (0,0,1,0,0)^T$$

$$U(1)_\phi \times \frac{S_F^2}{(\mathbb{Z}_2)_F}$$

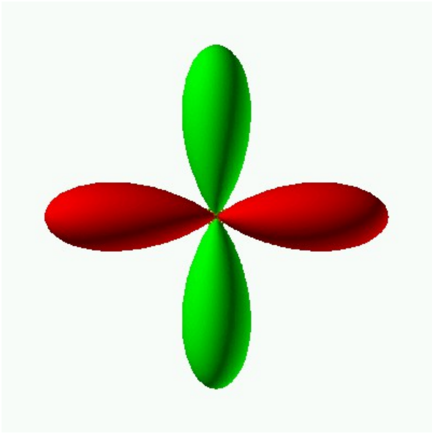


Biaxial Nematic:

$$\Psi_{\mathrm{B}} = (1,0,0,0,1)^T/\sqrt{2}$$

$$\frac{U(1)_\phi \times SO(3)_F}{(D_4)_{\phi+F}}$$

Two states are degenerate via another continuous degree of freedom

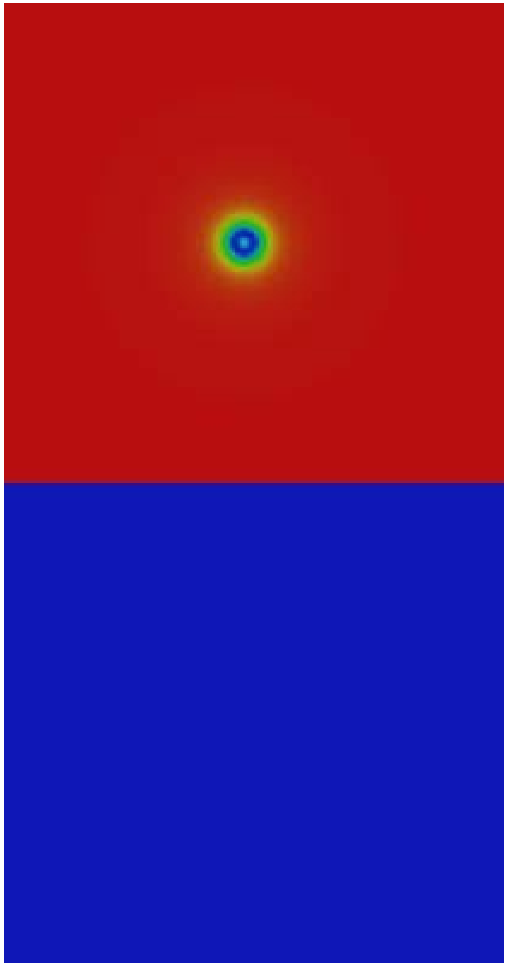


New order-parameter manifold

$$\frac{U(1)_\phi \times S_F^4}{(\mathbb{Z}_2)_{\phi+F}}$$

Quasi-Nambu-Goldstone current

S. Uchino, et al., PRL in press



Decay of vortex in biaxial nematic phase

Emission of quasi-Nambu-Goldstone current

Cyclic State vs. Singlet-trio Condensed State

For $c_1 > 0,\ c_2 > 0$

M. Koashi, and M. Ueda. PRL **84**, 1066 (2000)

Singlet-trio condensed state (only $\mathrm{U}(1)$ is broken)

$$|\Psi\rangle = \left[e^{i\varphi} \left(\frac{\sqrt{2}\hat{a}_0(\hat{a}_0^{\dagger 2} - 3a_1^\dagger a_{-1}^\dagger - 6a_2^\dagger a_{-2}^\dagger) + 3\sqrt{3}(a_1^{\dagger 2} a_{-2}^\dagger + a_{-1}^{\dagger 2} a_2^\dagger)}{\sqrt{35}} \right) \right]^{N/3} |0\rangle$$

Transition occurs under ~1μG

Cyclic state ($\mathrm{U}(1) \times \mathrm{SO}(3)$ is broken)

$$|\Psi\rangle = \left[\sum_m \Psi_m a_m^\dagger \right]^N |0\rangle$$

$$\Psi = e^{i\varphi} e^{-i\hat{\boldsymbol{F}}\cdot\boldsymbol{\alpha}} \begin{pmatrix} i/2 \\ 0 \\ 1/\sqrt{2} \\ 0 \\ i/2 \end{pmatrix}$$

Nematic State vs. Singlet-pair Condensed State

For $c_1 > 0,\ c_2 < 0$

M. Koashi, and M. Ueda. PRL **84**, 1066 (2000)

Singlet-pair condensed state (only $\mathrm{U}(1)$ is broken)

$$|\Psi\rangle = \left[e^{i\varphi} \left(\frac{\hat{a}_0^{\dagger 2} - 2a_1^\dagger a_{-1}^\dagger + a_2^\dagger a_{-2}^\dagger}{\sqrt{5}} \right) \right]^{N/2} |0\rangle$$

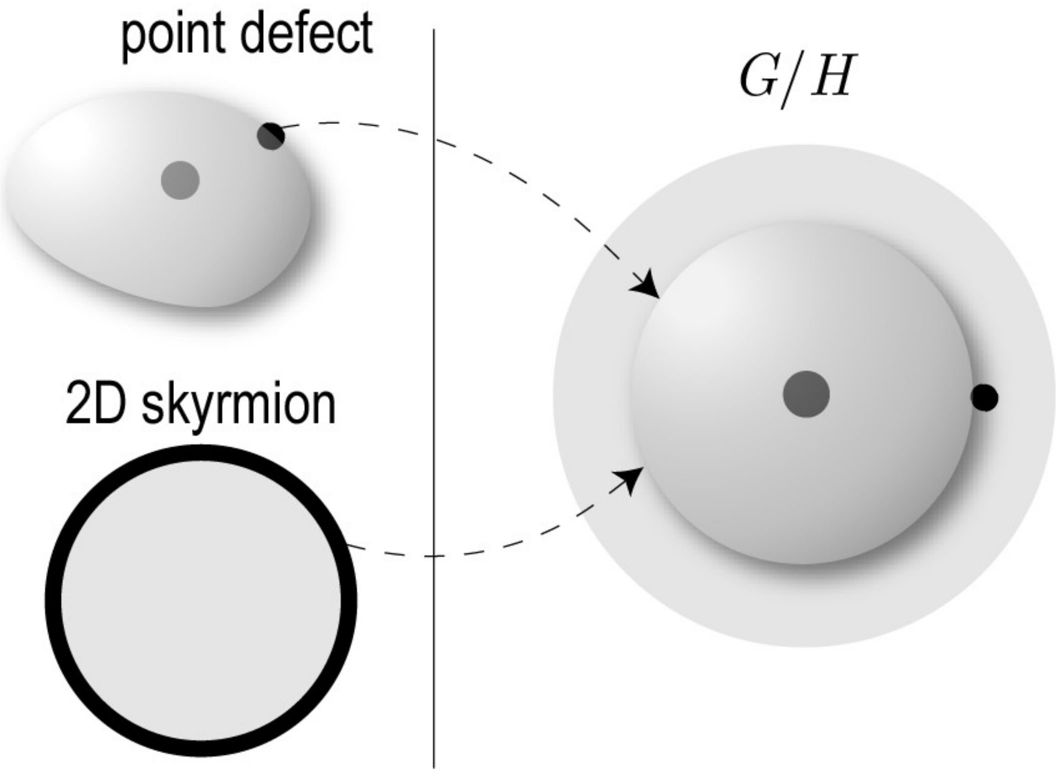
Transition occurs under ~1μG

Nematic state $(\mathrm{U}(1) \times \mathrm{SO}(3)$ is broken)

$$|\Psi\rangle = \left[\sum_m \Psi_m a_m^\dagger \right]^N |0\rangle \qquad \Psi = e^{i\varphi} e^{-i\hat{\boldsymbol{F}}\cdot\boldsymbol{\alpha}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} / \sqrt{2}$$

Topological defects (2nd homotopy group)

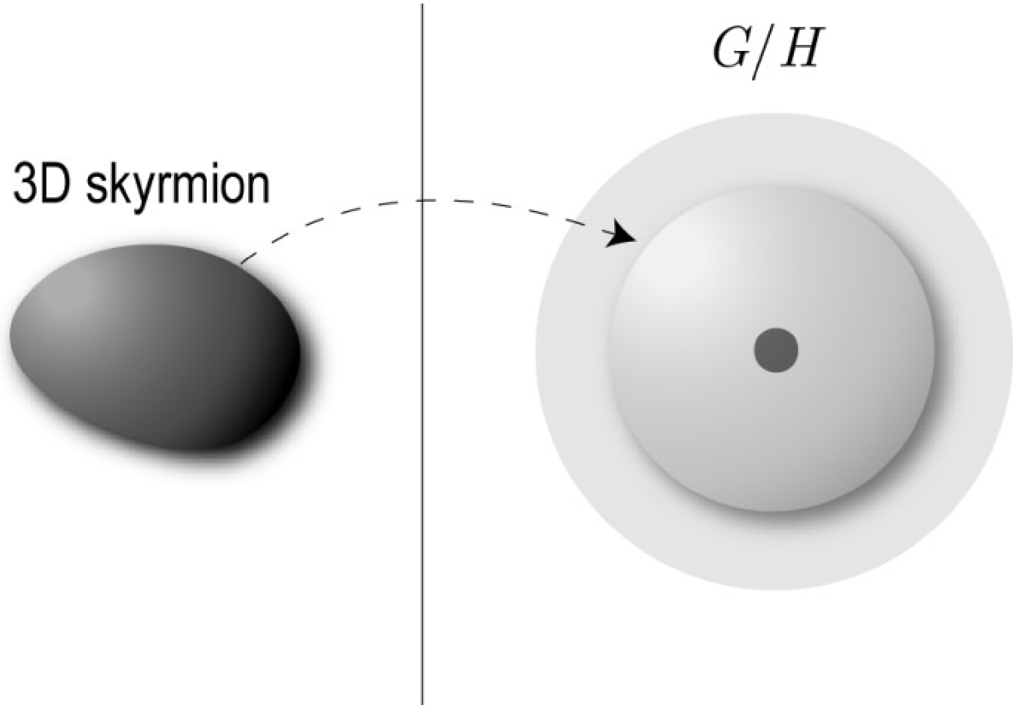
Point defect or 2D skyrmion (baby skyrmion)



Along the closed sphere, by how many times the sphere covers the singular point of Ψ (2nd homotopy group π_2)

Topological defects (3nd homotopy group)

3D skyrmion



Along the closed 3D sphere, by how many times the sphere covers the singular point of Ψ (3nd homotopy group π_3)