Non-Abelian Vortices in Spinor Bose-Einstein Condensates

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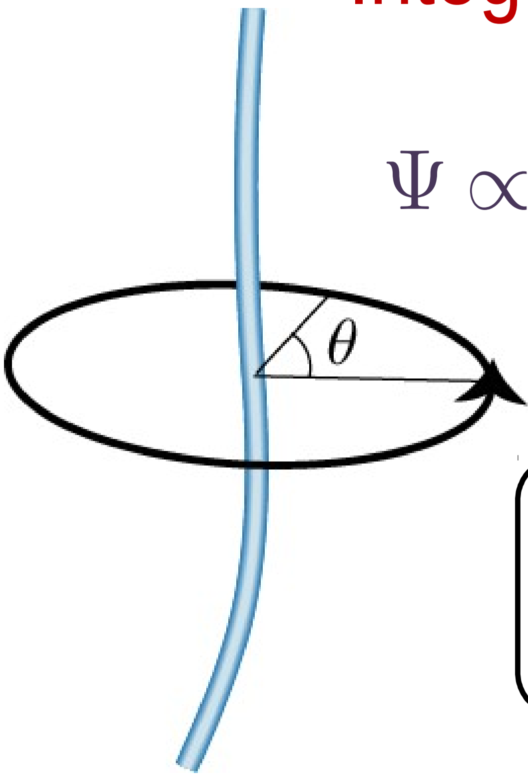
Conclusion

1. Non-Abelian vortices are realized in the cyclic phase of spin-2 spinor Bose-Einstein condensates.
2. Non-Abelian character becomes remarkable in collision dynamics of two vortices.
   I. We numerically show.
   II. We algebraically confirm.

Vortices in Bose-Einstein Condensates

Integer vortex (Scalar BEC or ^{4}He)



$\Psi \propto e^{in\theta}$

$\boldsymbol{v} = (n_{\text{ㄹE}}/m)\nabla\theta$: superfluid velocity

$\int \boldsymbol{v} \cdot d\boldsymbol{l} = nh/m$: circulation

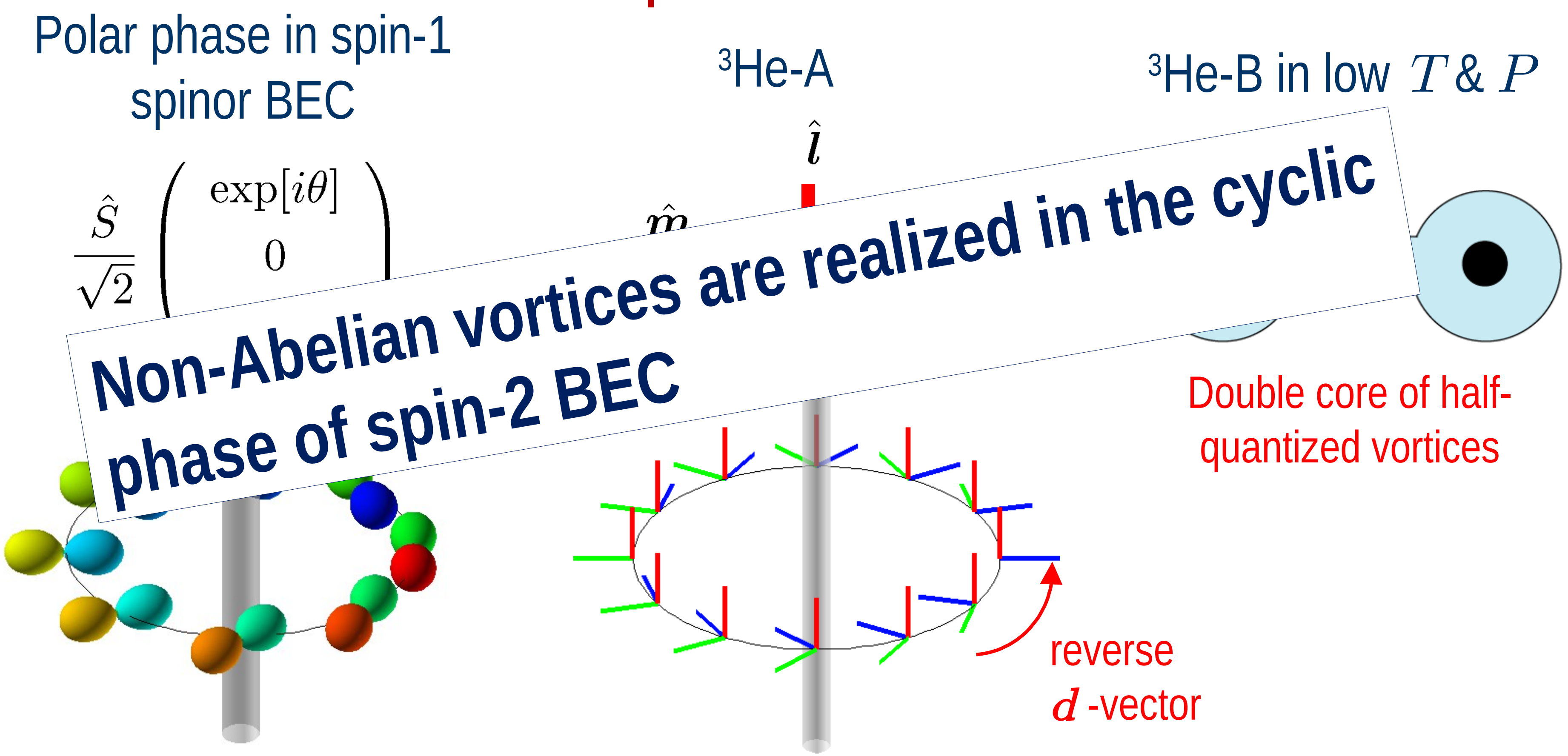
Around the vortex core

1. Phase winds by integer multiple of 2π.
2. Circulation takes integer multiple of h/m

Topological charge of vortex is characterized by additive group of integers $\longrightarrow$ Abelian vortex.

Vortices in Bose-Einstein Condensates

Half-quantized vortex

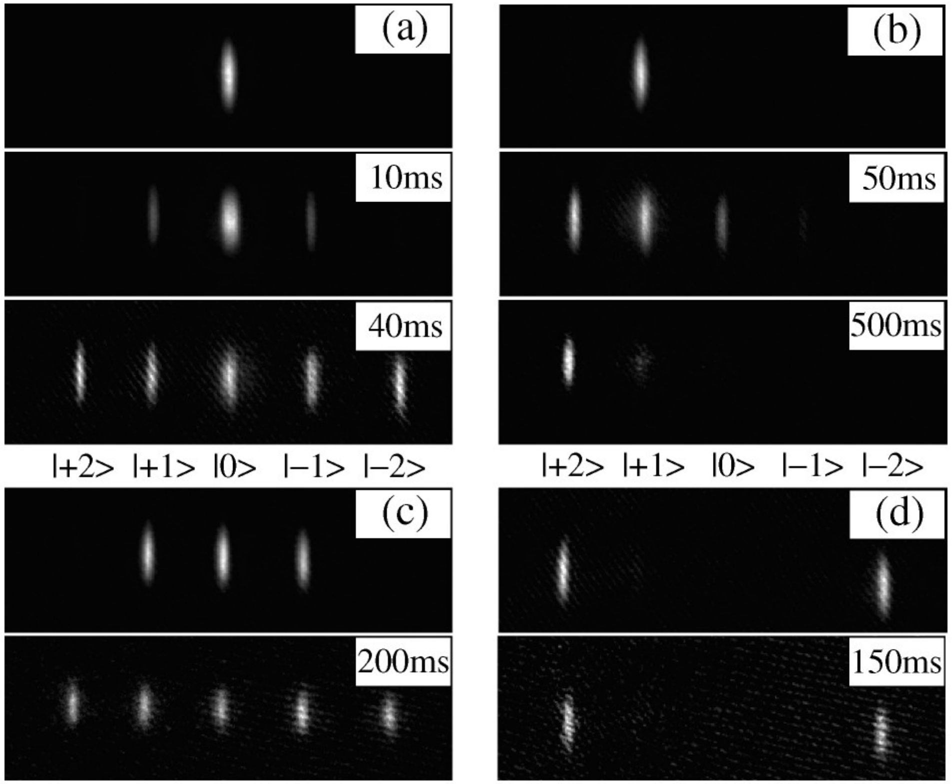


Topological charge is characterized by 2nd cyclic group

Spin-2 Spinor BEC

5 - component BEC : $\Psi = (\Psi_2, \Psi_1, \Psi_0, \Psi_{-1}, \Psi_{-2})^T$

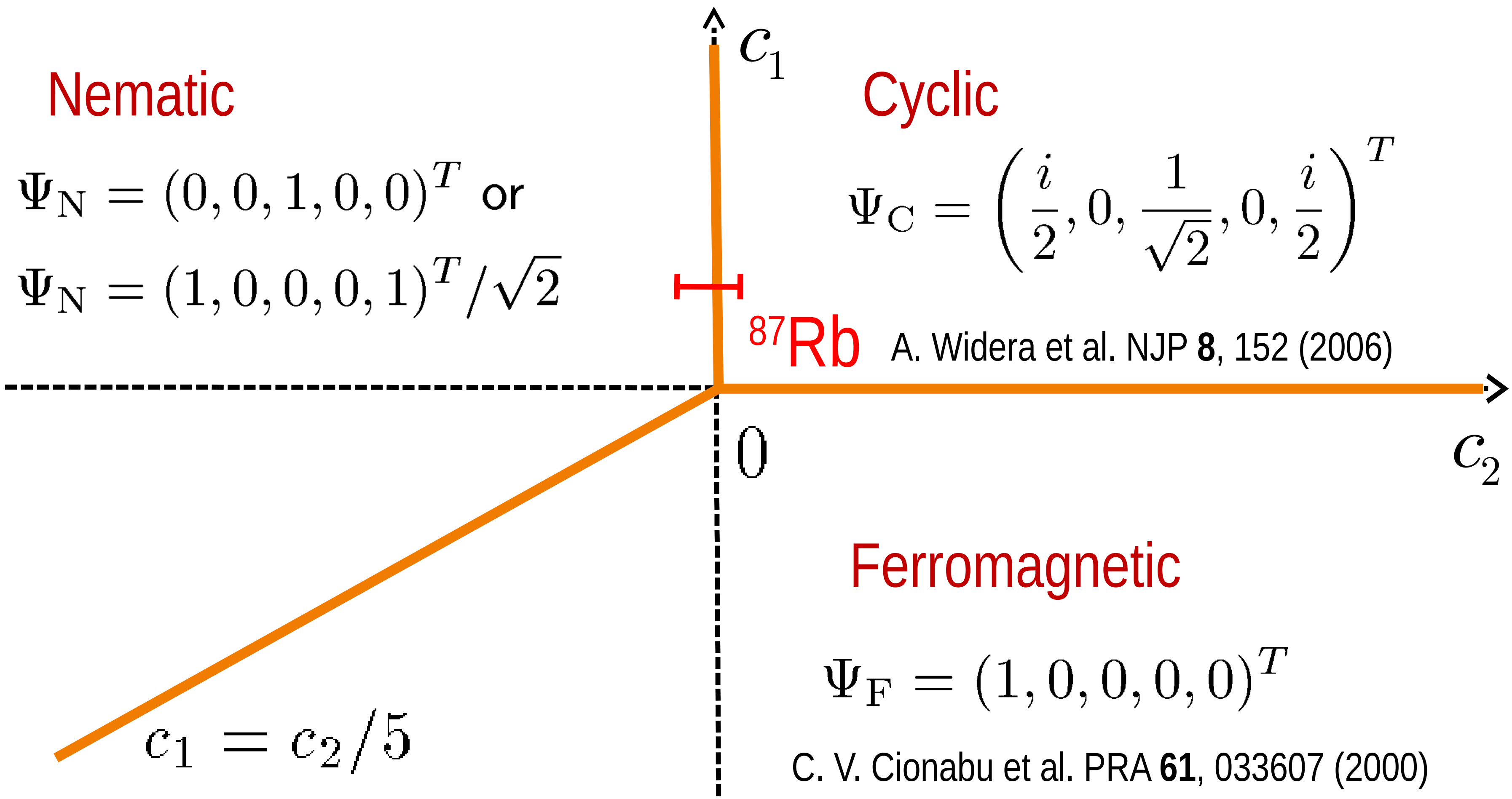
F = 2 ^{87}Rb BEC and its spin dynamics is observed



H. Schmaljohann et al. PRL **92**, 040402 (2004)

Ground State of Spin-2 Spinor BEC

$$c_0 = \frac{4\pi\hbar^2}{m}\frac{4a_2+3a_4}{7}, \quad c_1 = \frac{4\pi\hbar^2}{m}\frac{a_4-a_2}{7}, \quad c_2 = \frac{4\pi\hbar^2}{m}\frac{7a_0-10a_2+3a_4}{35}$$

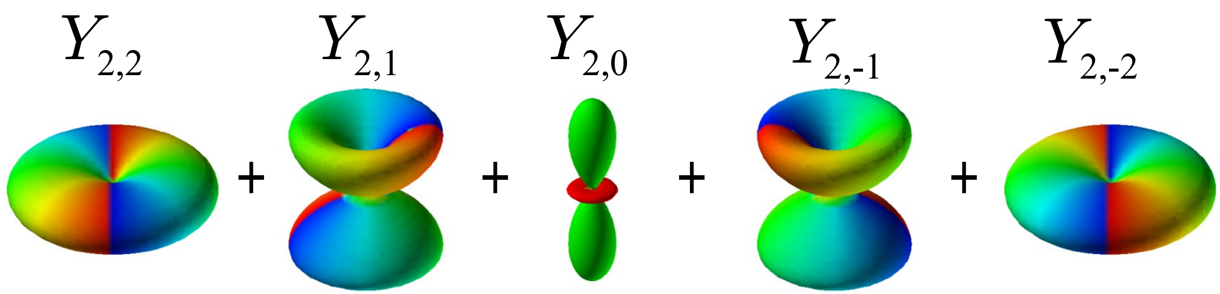


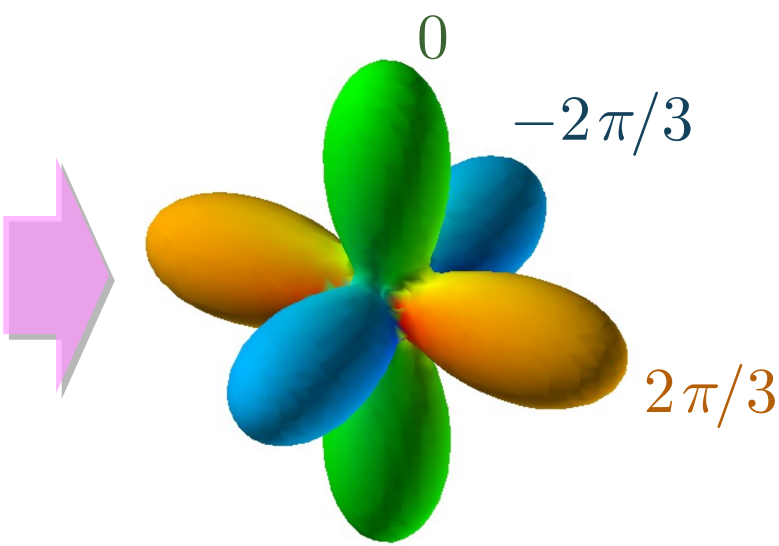
Spherical Harmonics Expression of Cyclic Phase

cyclic phase

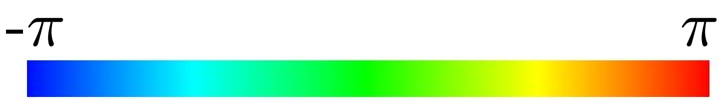
$$\Psi = \begin{pmatrix} i/2 \\ 0 \\ 1/\sqrt{2} \\ 0 \\ i/2 \end{pmatrix}$$

$$|\Psi\rangle = \sum_{m=-2}^{2} \Psi_m Y_{2,m}$$

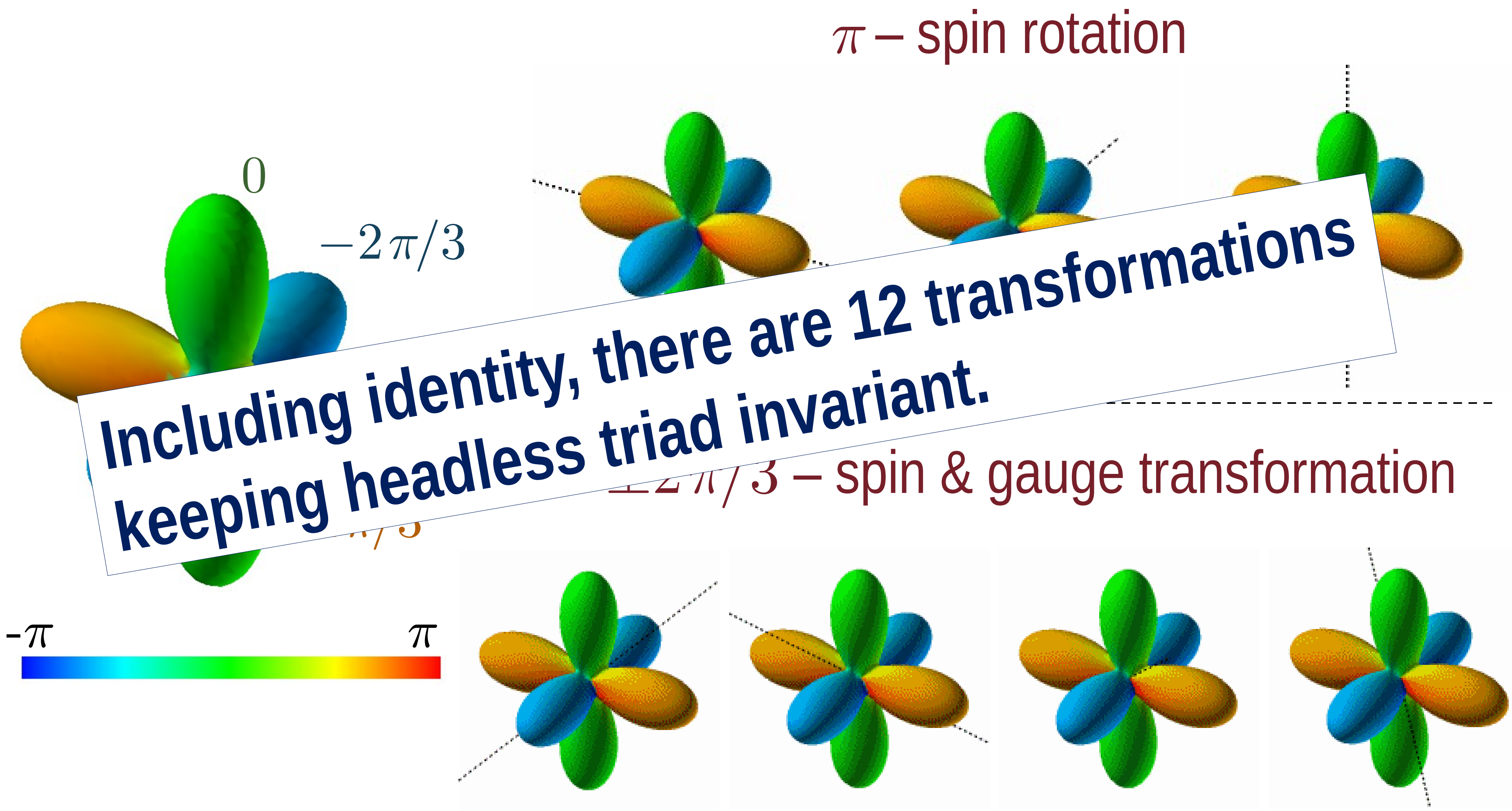




1. Cyclic state can be expressed as a headless triad
2. Phase difference between each lobe is $2\pi/3$



Invariant Spin or Spin – Gauge Transformation

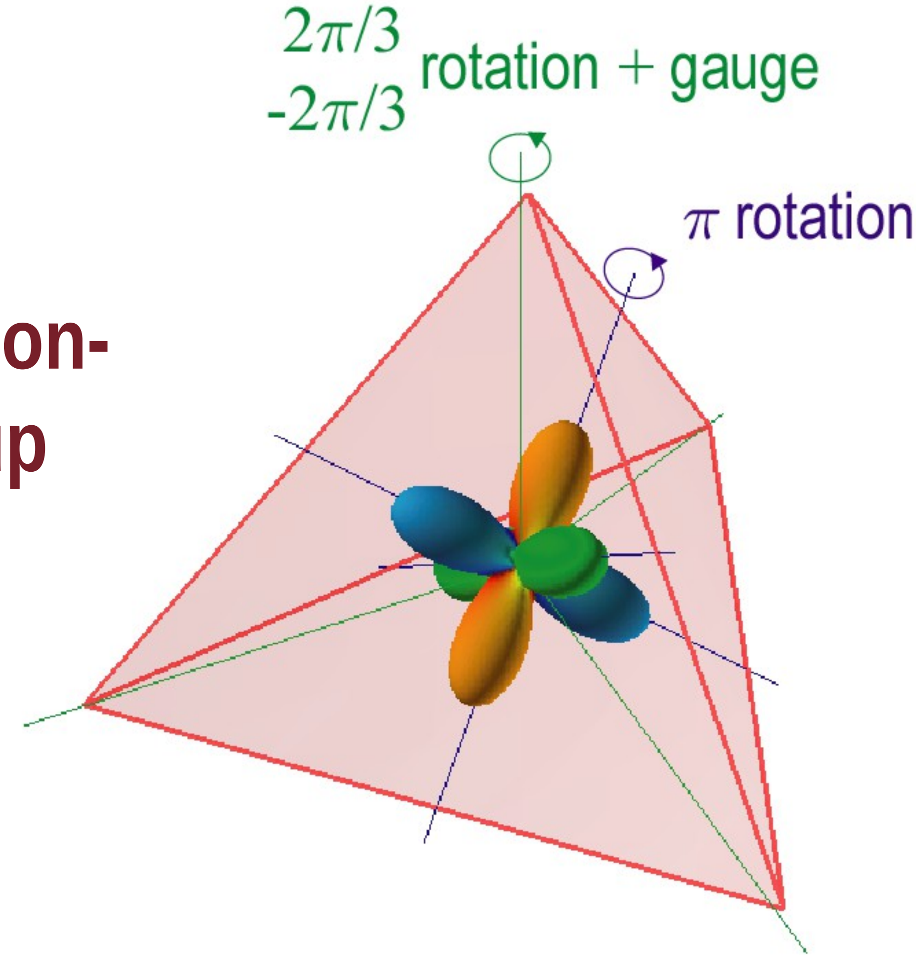


Including identity, there are 12 transformations keeping headless triad invariant.

H. Mäkelä et al. J. Phys. A **36**, 8555 (2003), G. W. Semenoff et al. PRL **98**, 100401 (2007)

Invariant Spin or Spin – Gauge Transformation

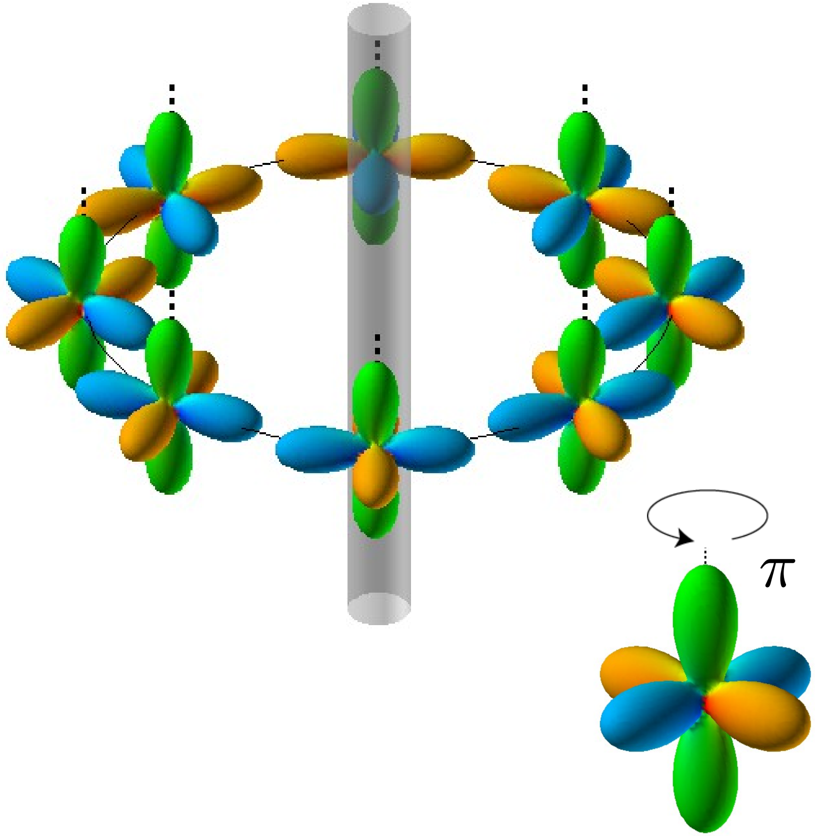
12 transformations form non-Abelian tetrahedral group



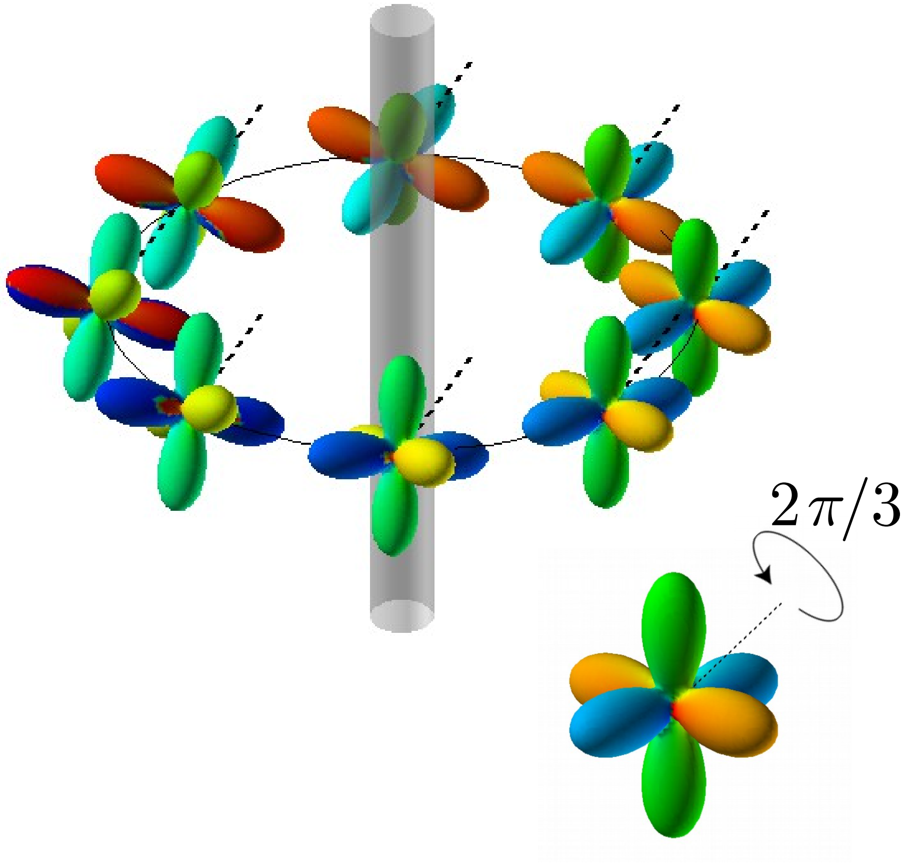
Vortices

Invariant transformations define vortices

1/2 spin vortex



1/3 vortex



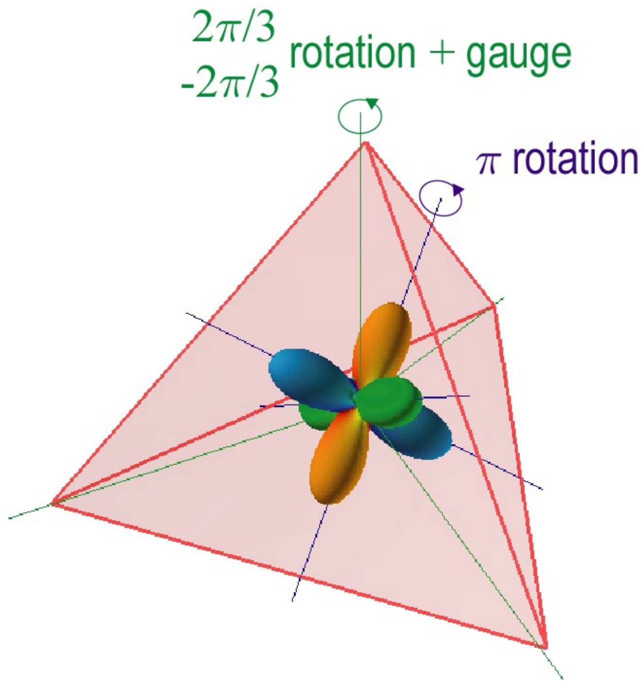
Topological Charge of Vortices

Scalar BEC

$$\Psi \propto e^{in\theta}$$

Topological charge :
Additive group of integer n
$\longrightarrow$ **Abelian vortices**

Cyclic phase in spin-2 spinor BEC



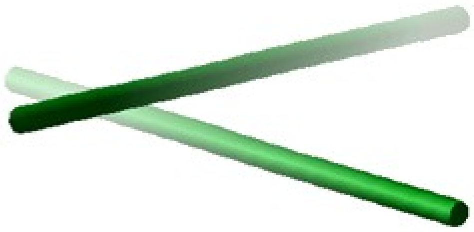
Topological charge : Tetrahedral group
$\longrightarrow$ **Non-Abelian vortices**

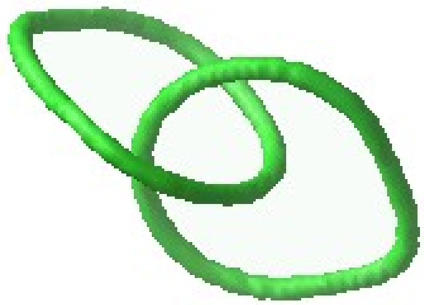
Collision Dynamics of Non-Abelian Vortices

Non-Abelian property of vortices becomes remarkable in their collision dynamics

→ Numerical simulation of Gross-Pitaevskii equation

Initial state : two straight vortices, linked vortex rings

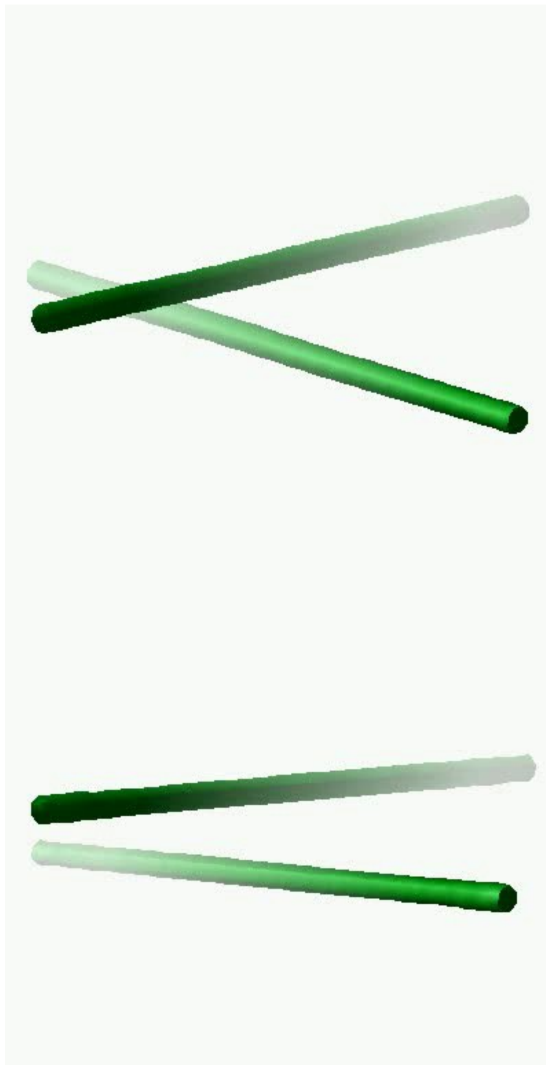




Collision of vortices with non-commutative charge forms a new "**rung**" vortex connecting two vortices

Same charge

Reconnection



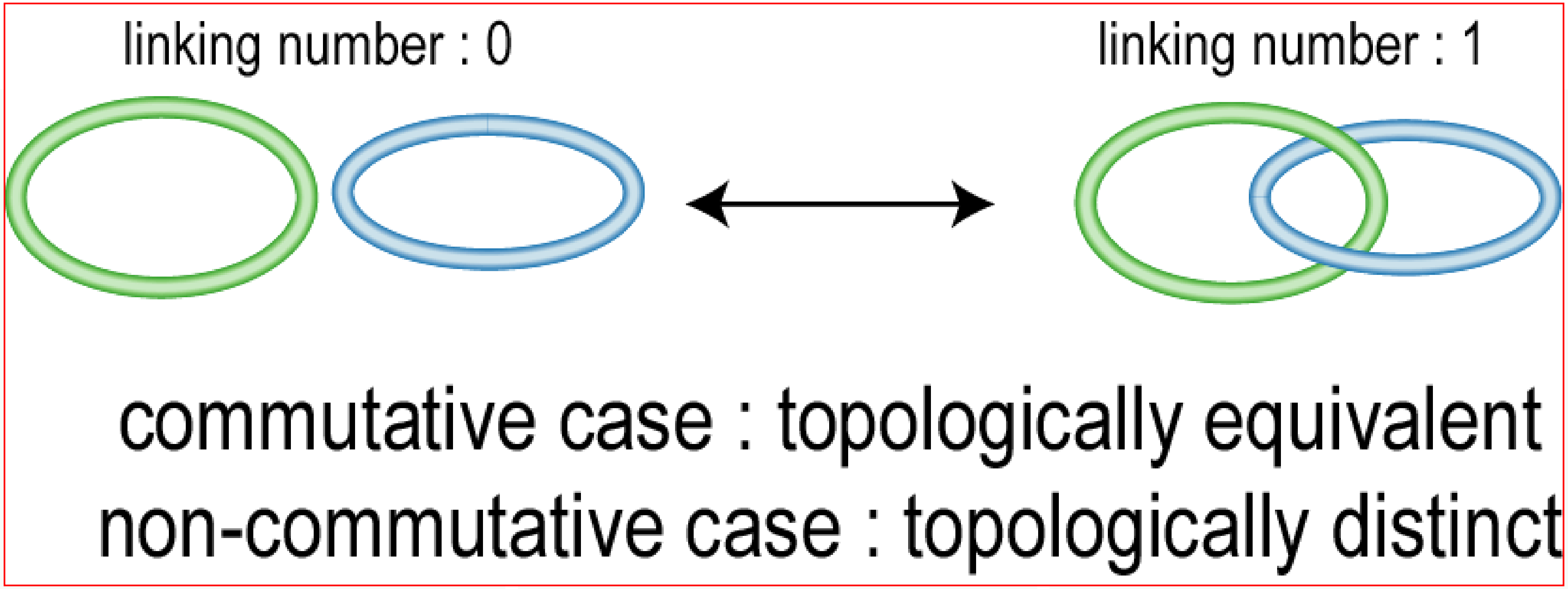
Commutative charges

Passing

Non-commutative charges

Rung vortex

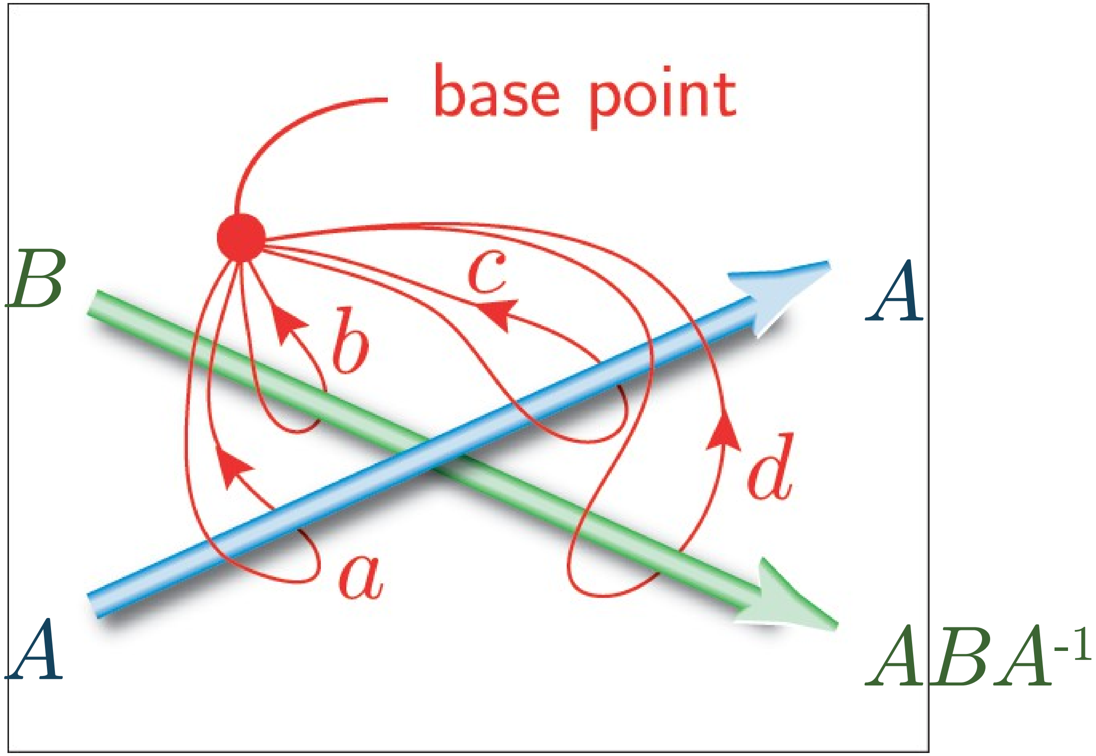
Collision Dynamics of Non-Abelian Vortices



Large ring | Unraveling of link | Rung vortex

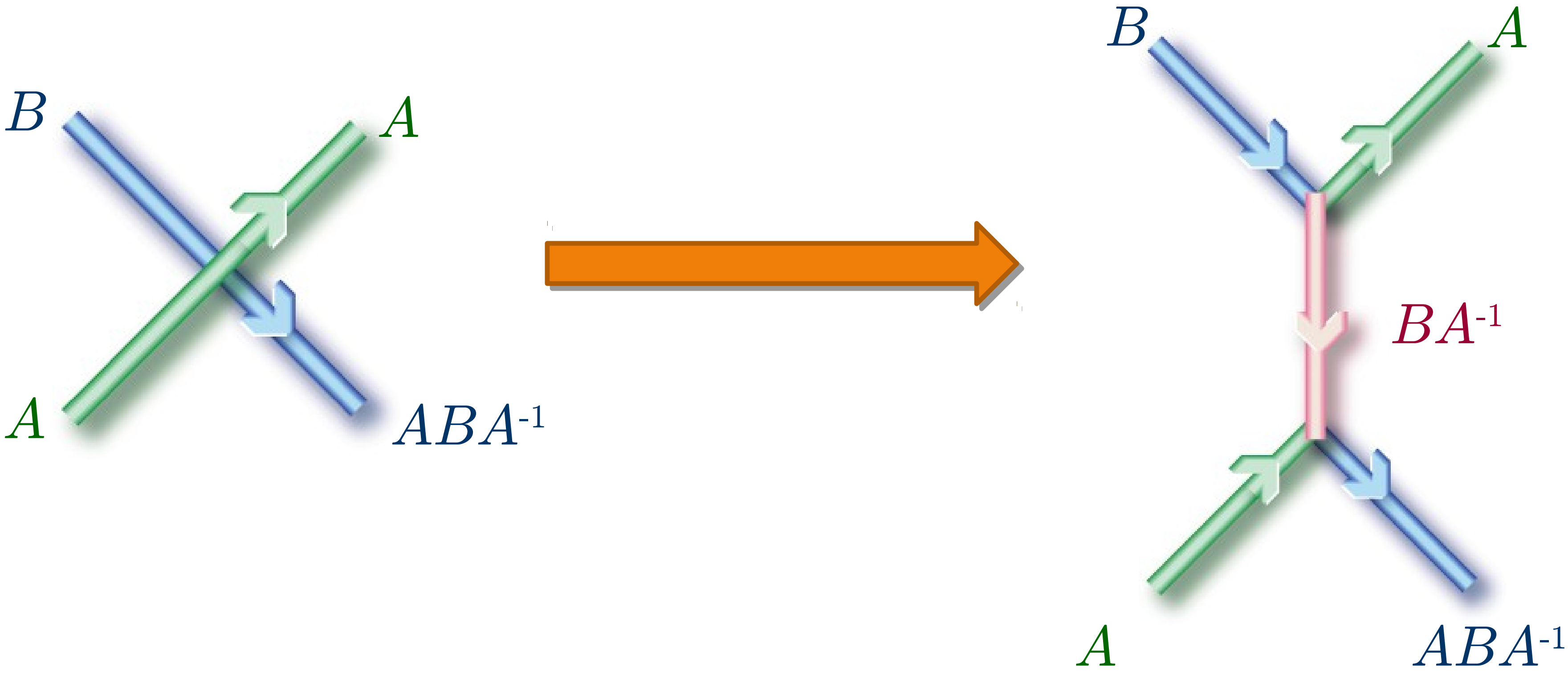
Linked vortices with non-commutative charges cannot unravel because of the formation of the rung vortex.

Algebra



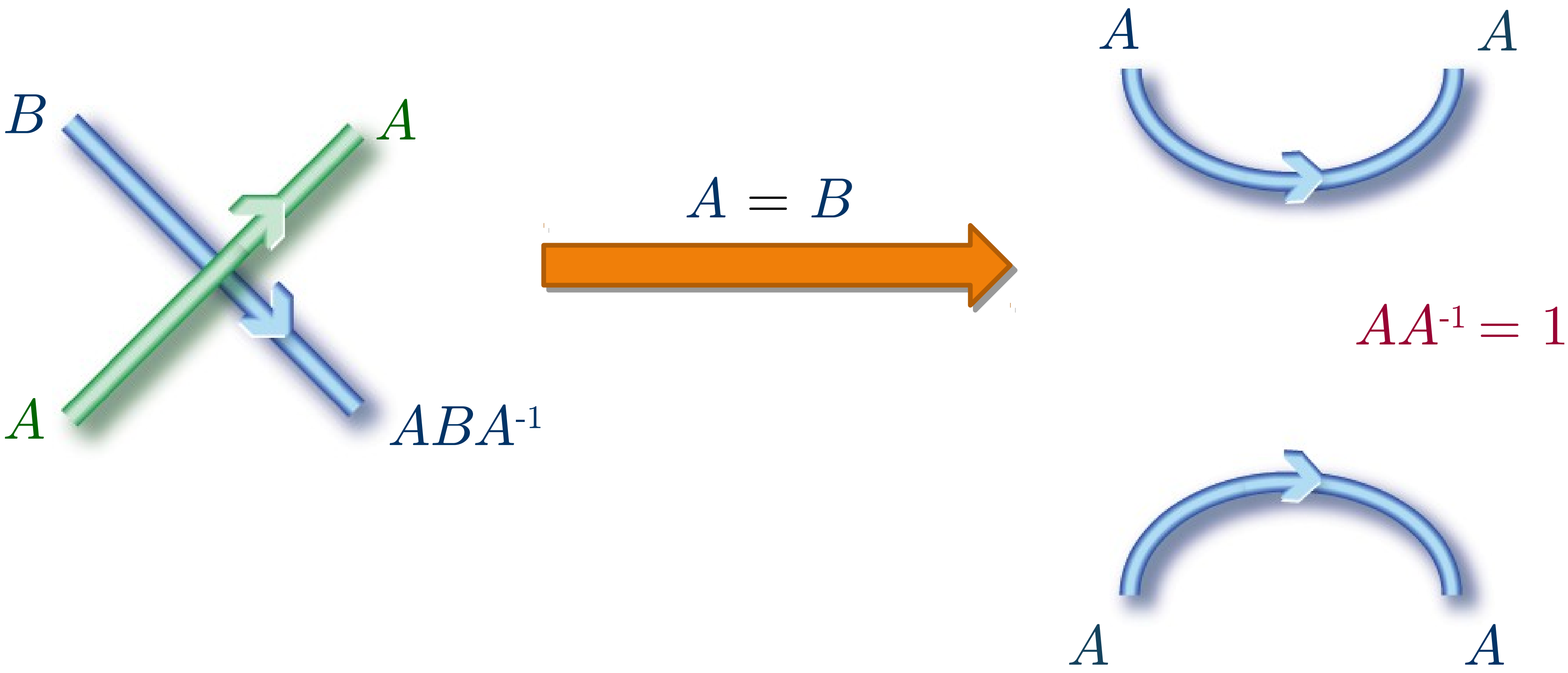
Topological charge of vortex can be fixed by a closed path encircling the vortex

Collision of Vortex



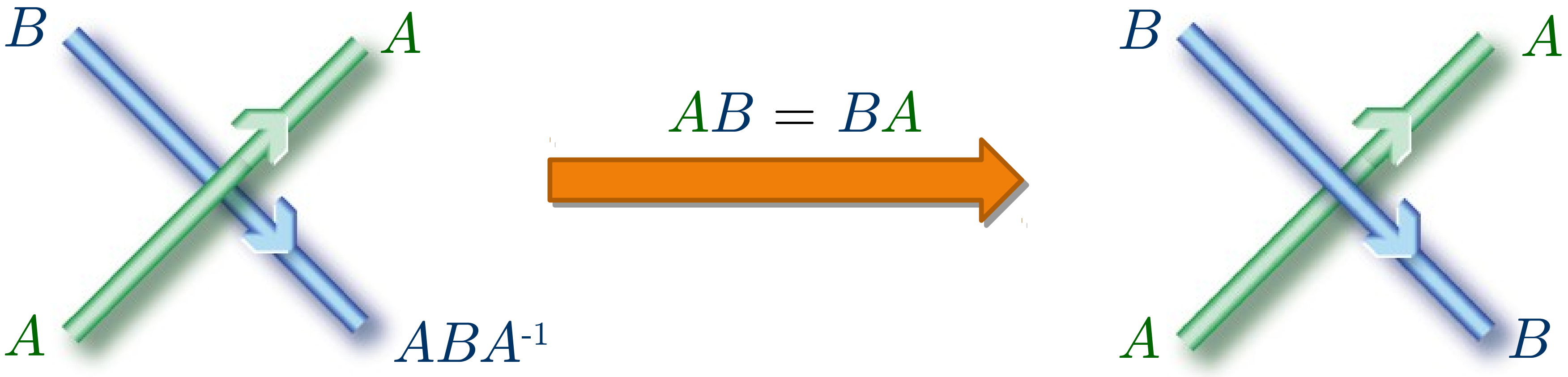
Rung BA^{-1} is formed through the collision.

Collision of Vortex



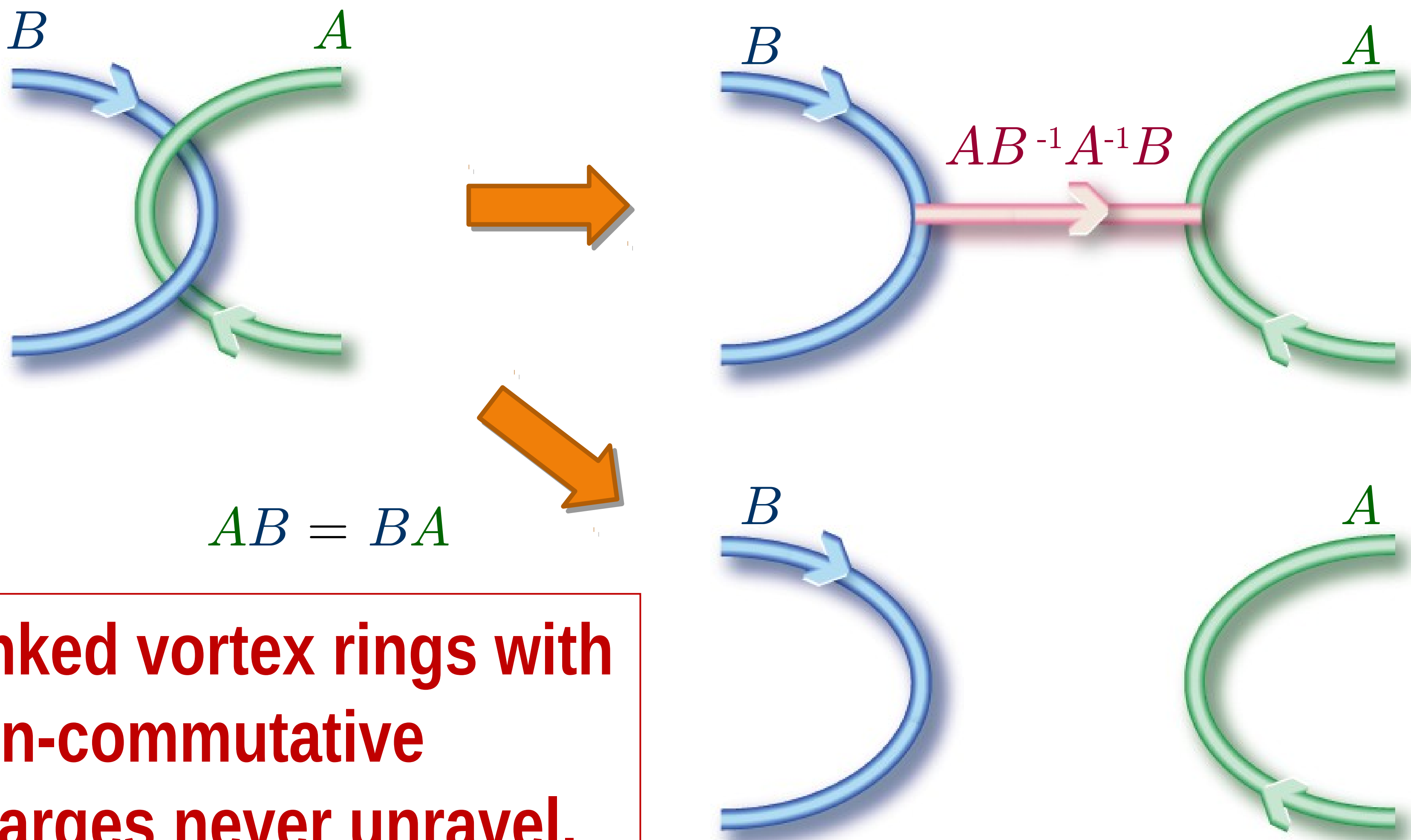
Rung disappears for the same charge resulting reconnection.

Collision of Vortex



Passing dynamics is also possible for commutative case

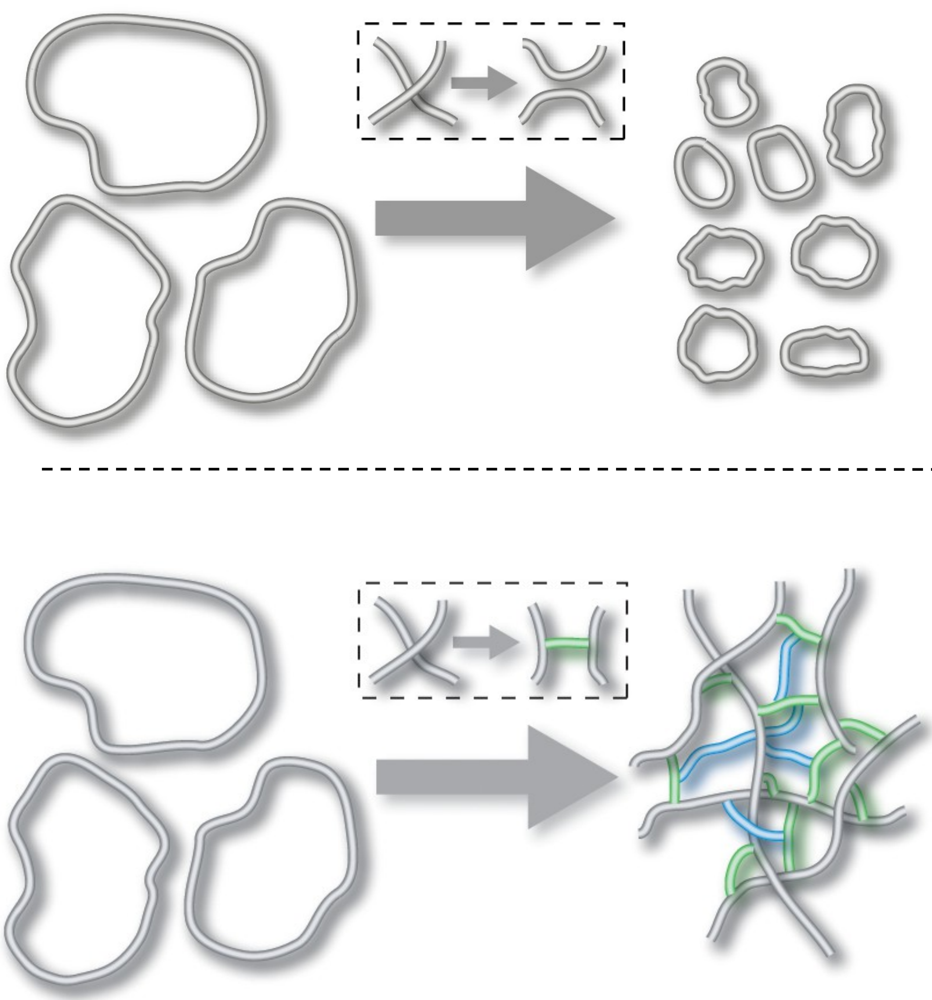
Linked Vortex Rings



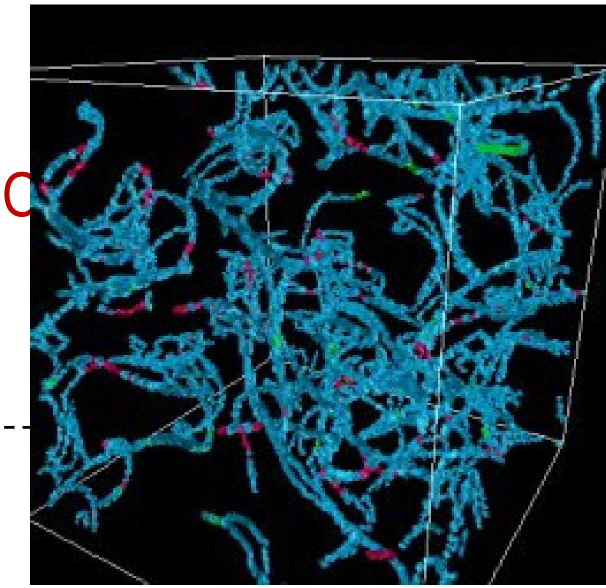
$AB = BA$

Linked vortex rings with non-commutative charges never unravel.

Application of Non-Abelian Vortices : N on-Abelian Turbulence



Preliminary simulation



↓

Large-scale networking structure of vortices through formation of rungs

New type of turbulence

Conclusion

1. Non-Abelian vortices are realized in the cyclic phase of spin-2 spinor Bose-Einstein condensates.
2. Non-Abelian character becomes remarkable in collision dynamics of two vortices.
   I. Rung vortex is formed after the collision.
   II. Linked vortex rings never unravel

M. Kobayashi, Y. Kawaguchi, M. Nitta, and M. Ueda. PRL **103**, 115301(2009)

Cyclic State vs. Singlet-trio Condensed State

For $c_1 > 0,\ c_2 > 0$

M. Koashi, and M. Ueda. PRL **84**, 1066 (2000)

Singlet-trio condensed state (only $\mathrm{U}(1)$ is broken)

$$|\Psi\rangle = \left[e^{i\varphi} \left(\frac{\sqrt{2}\hat{a}_0(\hat{a}_0^{\dagger 2} - 3a_1^\dagger a_{-1}^\dagger - 6a_2^\dagger a_{-2}^\dagger) + 3\sqrt{3}(a_1^{\dagger 2} a_{-2}^\dagger + a_{-1}^{\dagger 2} a_2^\dagger)}{\sqrt{35}} \right) \right]^{N/3} |0\rangle$$

Transition occurs under ~1μG

Cyclic state ($\mathrm{U}(1) \times \mathrm{SO}(3)$ is broken)

$$|\Psi\rangle = \left[\sum_m \Psi_m a_m^\dagger \right]^N |0\rangle$$

$$\Psi = e^{i\varphi} e^{-i\hat{\boldsymbol{F}}\cdot\boldsymbol{\alpha}} \begin{pmatrix} i/2 \\ 0 \\ 1/\sqrt{2} \\ 0 \\ i/2 \end{pmatrix}$$

Nematic State vs. Singlet-pair Condensed State

For $c_1>0,\ c_2<0$

M. Koashi, and M. Ueda. PRL **84**, 1066 (2000)

Singlet-pair condensed state (only $\mathrm{U}(1)$ is broken)

$$|\Psi\rangle = \left[e^{i\varphi} \left(\frac{\hat{a}_0^{\dagger 2} - 2a_1^\dagger a_{-1}^\dagger + a_2^\dagger a_{-2}^\dagger}{\sqrt{5}} \right) \right]^{N/2} |0\rangle$$

Transition occurs under ~1μG

Nematic state $(\mathrm{U}(1)\ \times\ \mathrm{SO}(3)$ is broken)

$$|\Psi\rangle = \left[\sum_m \Psi_m a_m^\dagger \right]^N |0\rangle \qquad \Psi = e^{i\varphi} e^{-i\hat{\boldsymbol{F}}\cdot\boldsymbol{\alpha}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} / \sqrt{2}$$

Hamiltonian of Spin-2 Spinor BEC

Bose system with spin degrees of freedom

$$H = \int d\boldsymbol{x} \sum_m \Psi_m^\dagger(\boldsymbol{x}) \left(-\frac{\hbar^2}{2M}\nabla^2 + V(\boldsymbol{x}) \right) \Psi_m(\boldsymbol{x})$$

$$+ \frac{1}{2} \int d\boldsymbol{x}_1 \int d\boldsymbol{x}_2 \sum_{m_1, m_2, m_1', m_2'} \Psi_{m_1}^\dagger(\boldsymbol{x}_1) \Psi_{m_2}^\dagger(\boldsymbol{x}_2) V_{m_1 m_2}^{m_1' m_2'}(\boldsymbol{x}_1 - \boldsymbol{x}_2) \Psi_{m_2'}(\boldsymbol{x}_2) \Psi_{m_1'}(\boldsymbol{x}_1)$$

Low energy contact interaction $(l = 0)$

$$V_{m_1 m_2}^{m_1' m_2'}(\boldsymbol{x}_1 - \boldsymbol{x}_2) = \delta(\boldsymbol{x}_1 - \boldsymbol{x}_2) \sum_{F=\mathrm{even}} g_F \sum_M \langle m1, m2 | F, M \rangle \langle F, M | m_1' m_2' \rangle$$

$\langle m_1 m_2 | F, M \rangle$: Clebsch-Gordan coefficient

Hamiltonian of Spin-2 Spinor BEC

Mean-field approximation

$$|\Psi\rangle = \left[\sum_m \Psi_m a_m^\dagger\right]^N |0\rangle$$

$$\langle H\rangle = \int d\boldsymbol{x} \left[\frac{\hbar^2}{2M}\sum_m |\nabla\Psi_m|^2 + \frac{c_0}{2}\rho^2 + \frac{c_1}{2}\boldsymbol{F}^2 + \frac{c_2}{2}|A_{20}|^2\right]$$

density

spin density

singlet – pair amplitude

$$\rho(\boldsymbol{x}) = \sum_m \Psi_m^*(\boldsymbol{x})\Psi_m(\boldsymbol{x})$$

$$\boldsymbol{F}(\boldsymbol{x}) = \sum_{m,m'} \Psi_m^*(\boldsymbol{x})\hat{\boldsymbol{F}}_{mm'}(\boldsymbol{x})\Psi_{m'}(\boldsymbol{x})$$

$$A_{20}(\boldsymbol{x}) = \sum_m (-1)^m \Psi_m(\boldsymbol{x})\Psi_{-m}(\boldsymbol{x})$$

$$c_0 = \frac{4\pi\hbar^2}{m}\frac{4a_2+3a_4}{7},$$

$$c_1 = \frac{4\pi\hbar^2}{m}\frac{a_4-a_2}{7},$$

$$c_2 = \frac{4\pi\hbar^2}{m}\frac{7a_0-10a_2+3a_4}{35}$$

Breaking of $U(1)_{\mathrm{G}} \times SO(3)_{\mathrm{S}}$

$$\langle H \rangle = \int d\boldsymbol{x} \left[\frac{\hbar^2}{2M} \sum_m |\nabla \Psi_m|^2 + \frac{c_0}{2}\rho^2 + \frac{c_1}{2}\boldsymbol{F}^2 + \frac{c_2}{2}|A_{20}|^2 \right]$$

$$\begin{pmatrix} \Psi_2 \\ \Psi_1 \\ \Psi_0 \\ \Psi_{-1} \\ \Psi_{-2} \end{pmatrix} = e^{i\varphi} \, e^{-i\hat{\boldsymbol{F}}\cdot\boldsymbol{\alpha}} \, \Psi_{\mathrm{Base}}$$

Ψ_{Base} ← Fixed from Hamiltonian

$e^{i\varphi}$: Gauge transformation： $U(1)_{\mathrm{G}}$

$e^{-i\hat{\boldsymbol{F}}\cdot\boldsymbol{\alpha}}$: Spin rotation： $SO(3)_{\mathrm{S}}$

Gross-Pitaevskii Equation

$$i\hbar\frac{\partial\Psi_m}{\partial t}=\frac{\delta H}{\delta\Psi_m^*}$$

$$i\hbar\frac{\partial\Psi_2}{\partial t}=-\frac{\hbar^2}{2M}\nabla^2\Psi_2+c_0\rho\Psi_2+c_1(F_-\Psi_1+2F_z\Psi_2)+c_2A_{20}\Psi^*_{-2}$$

$$i\hbar\frac{\partial\Psi_1}{\partial t}=-\frac{\hbar^2}{2M}\nabla^2\Psi_1+c_0\rho\Psi_1+c_1\left(\frac{\sqrt{6}}{2}F_-\Psi_0+F_+\Psi_2+F_z\Psi_1\right)-c_2A_{20}\Psi^*_{-1}$$

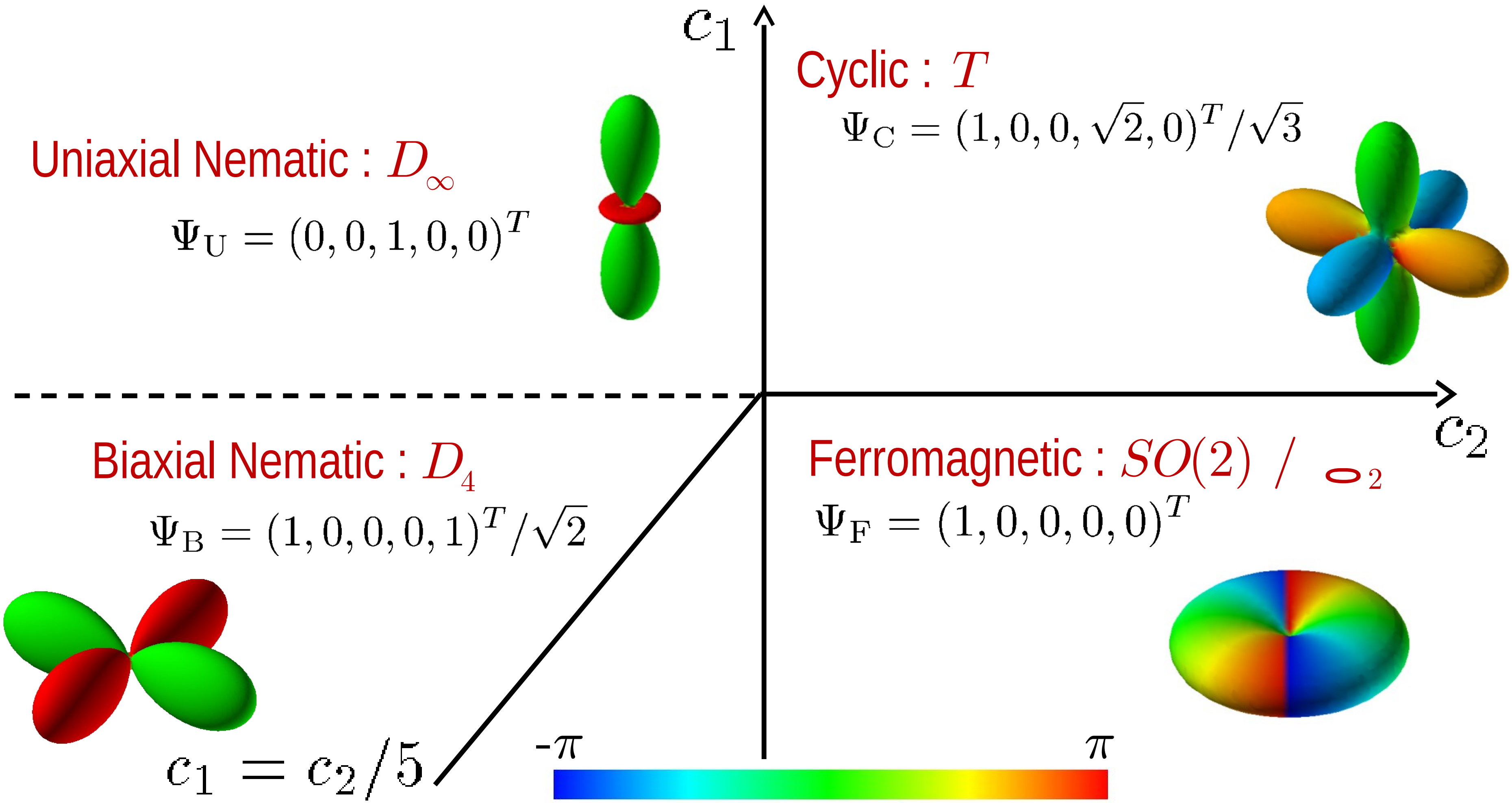
$$i\hbar\frac{\partial\Psi_0}{\partial t}=-\frac{\hbar^2}{2M}\nabla^2\Psi_0+c_0\rho\Psi_0+\frac{\sqrt{6}}{2}c_1(F_-\Psi_{-1}+F_+\Psi_1)+c_2A_{20}\Psi^*_0$$

$$i\hbar\frac{\partial\Psi_{-1}}{\partial t}=-\frac{\hbar^2}{2M}\nabla^2\Psi_{-1}+c_0\rho\Psi_{-1}+c_1\left(\frac{\sqrt{6}}{2}F_+\Psi_0+F_-\Psi_{-2}-F_z\Psi_{-1}\right)-c_2A_{20}\Psi^*_1$$

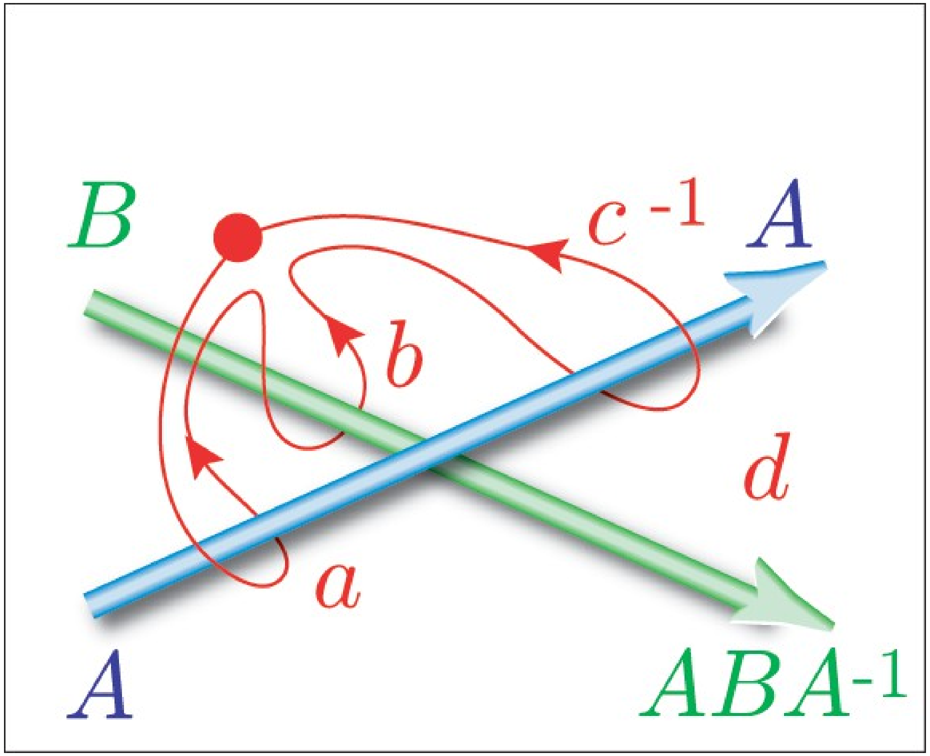
$$i\hbar\frac{\partial\Psi_{-2}}{\partial t}=-\frac{\hbar^2}{2M}\nabla^2\Psi_{-2}+c_0\rho\Psi_{-2}+c_1(F_+\Psi_{-1}-2F_z\Psi_{-2})+c_2A_{20}\Psi^*_2$$

Ground State Phase Diagram

S. Uchino, M. Kobayashi, and M. Ueda. PRA **81**, 063632 (2010)

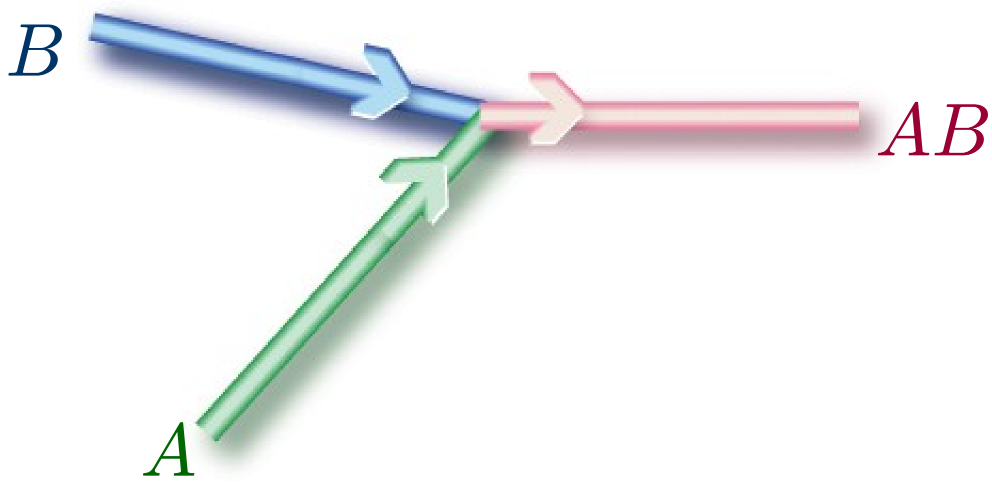


Algebra

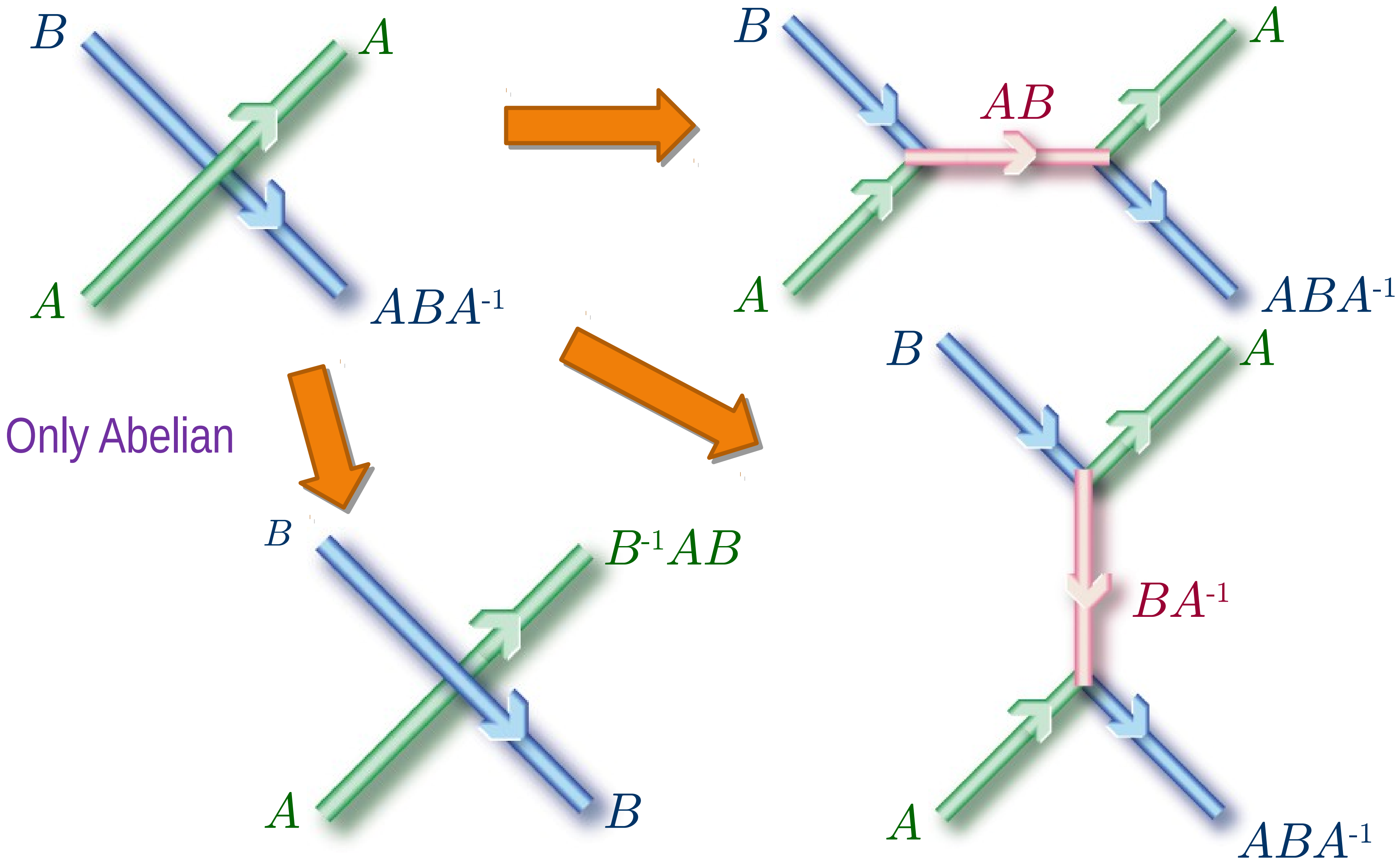


Path d defines vortex B as ABA^{-1}
(same conjugacy class)

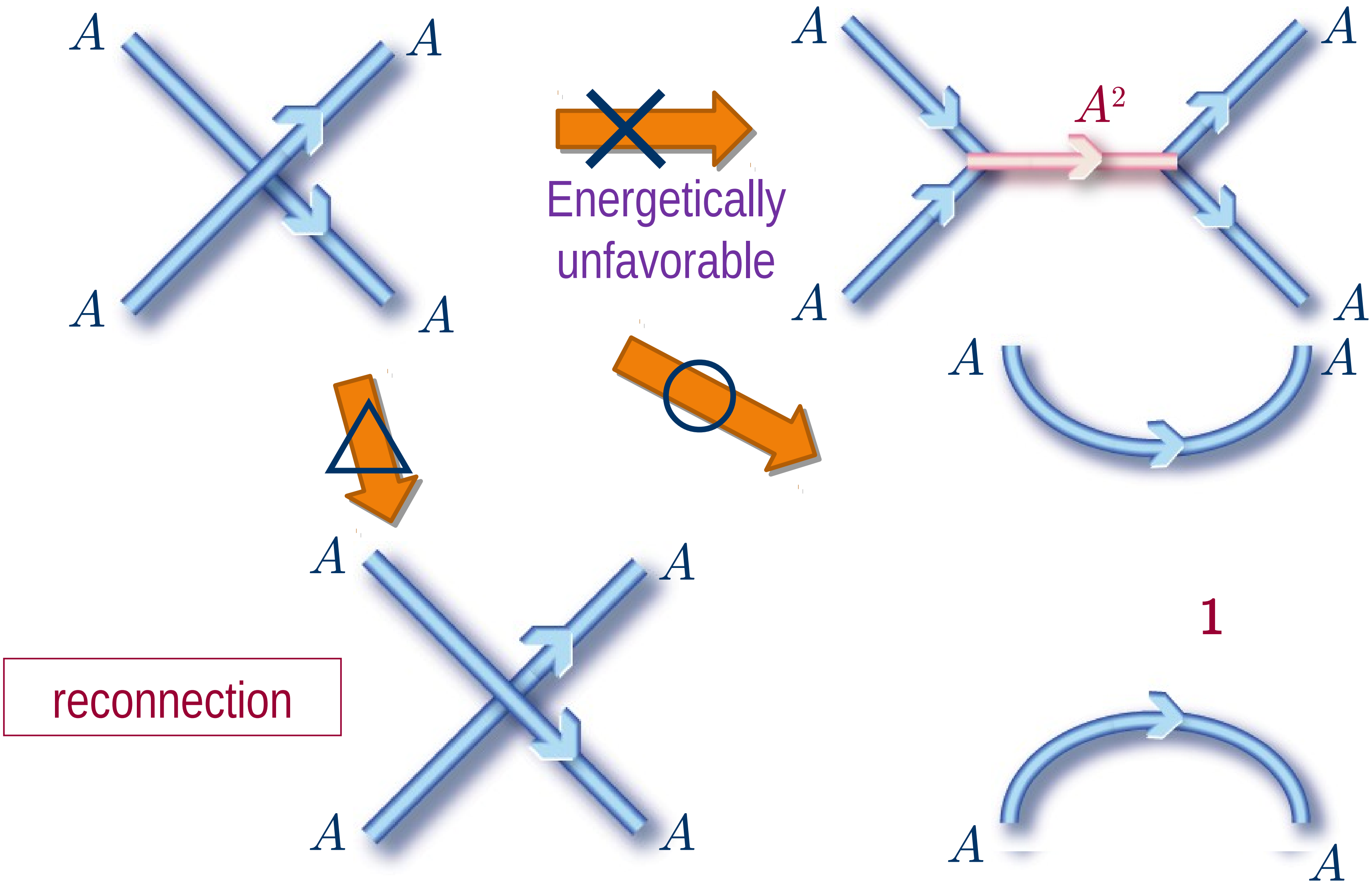
Y – Shaped Structure



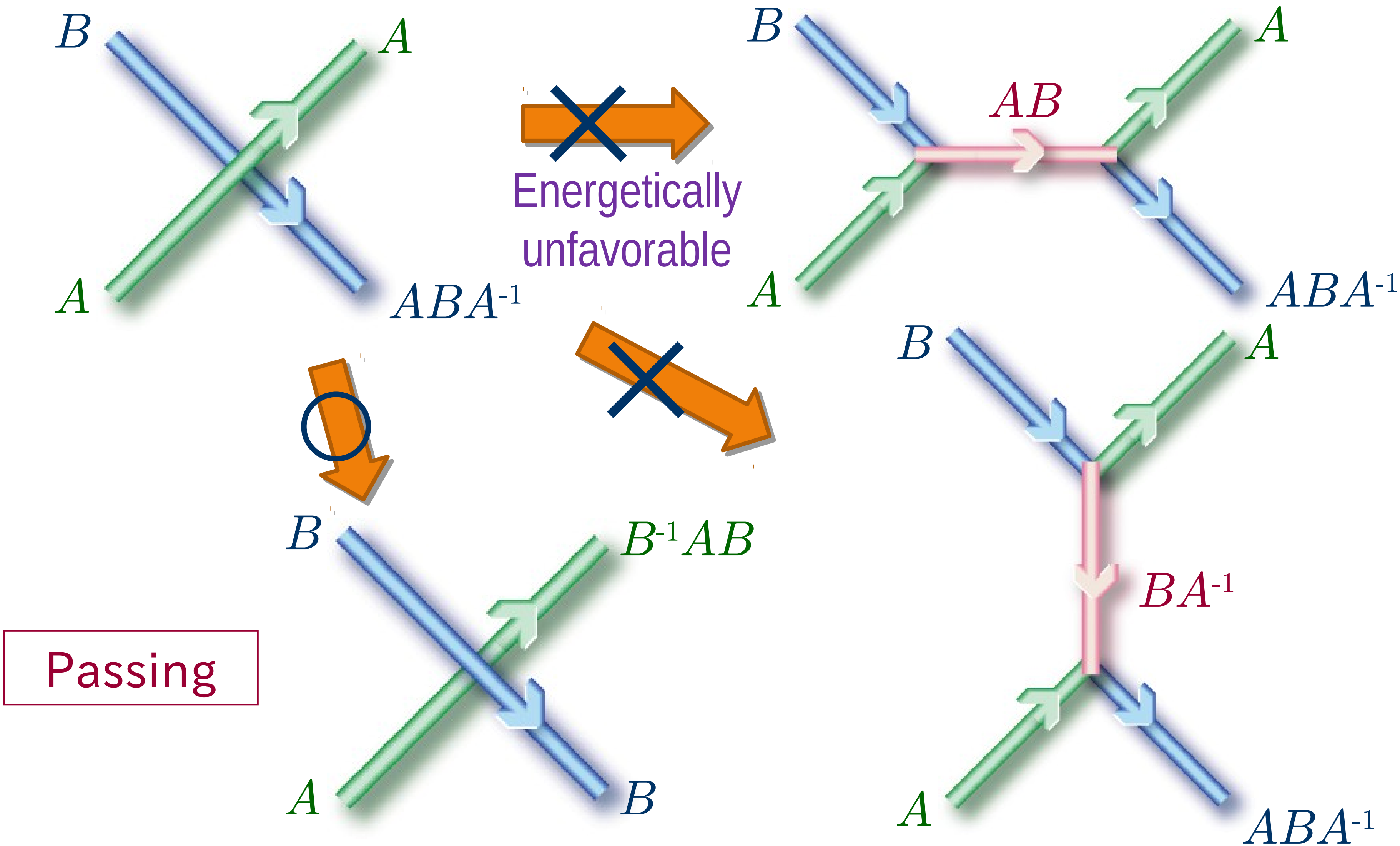
Collision of Vortex



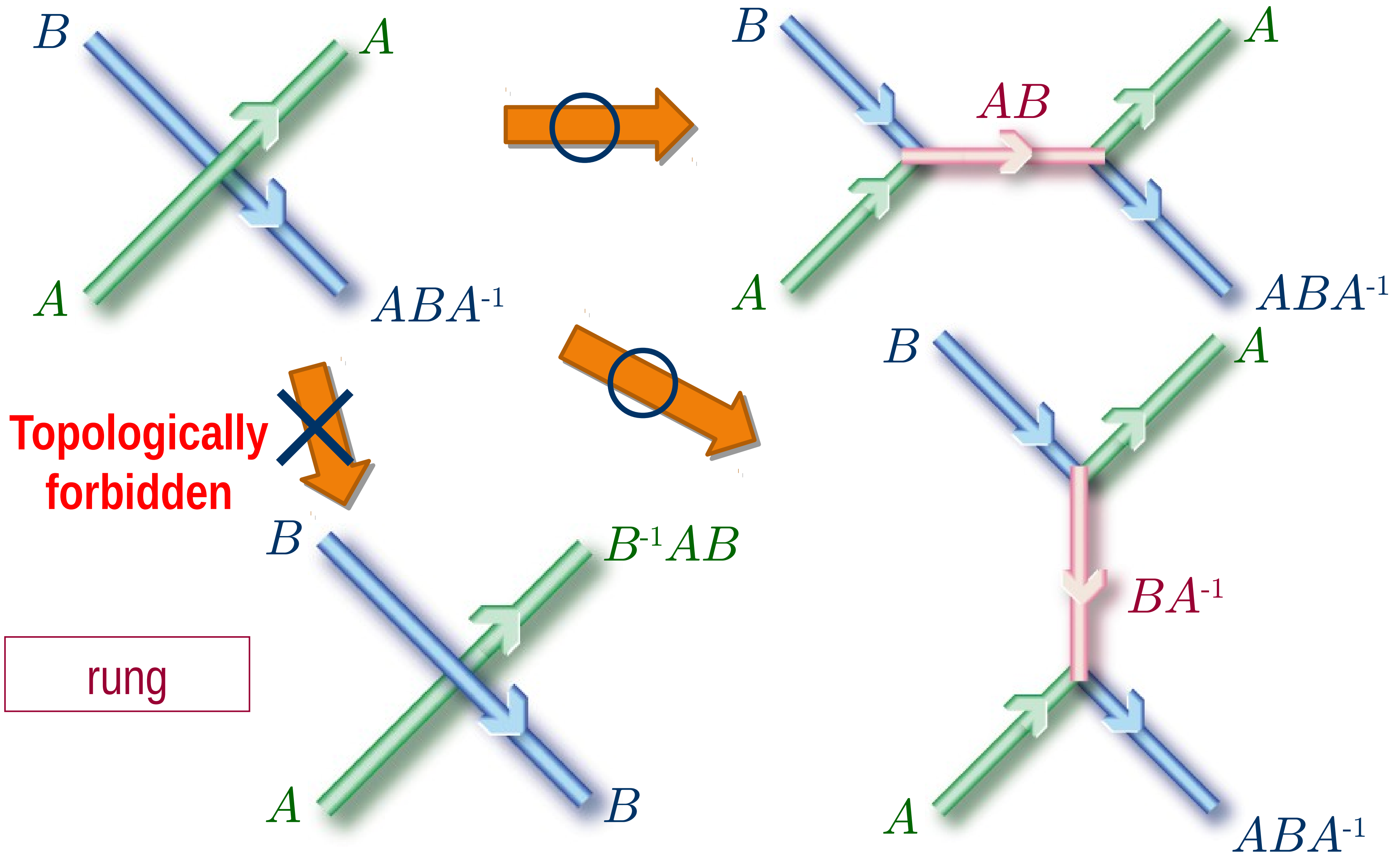
Same Charge


A
A
A
A
Energetically
unfavorable
A
A
A²
A
A
A
A
1
A
A
reconnection
A
A
A
A

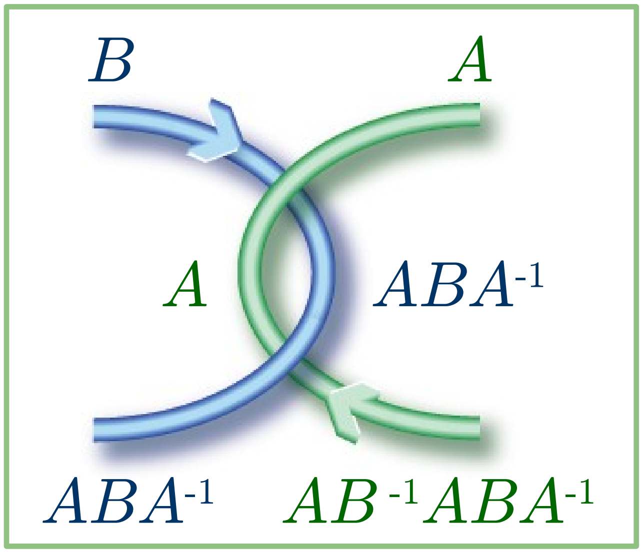
Commutative Charge

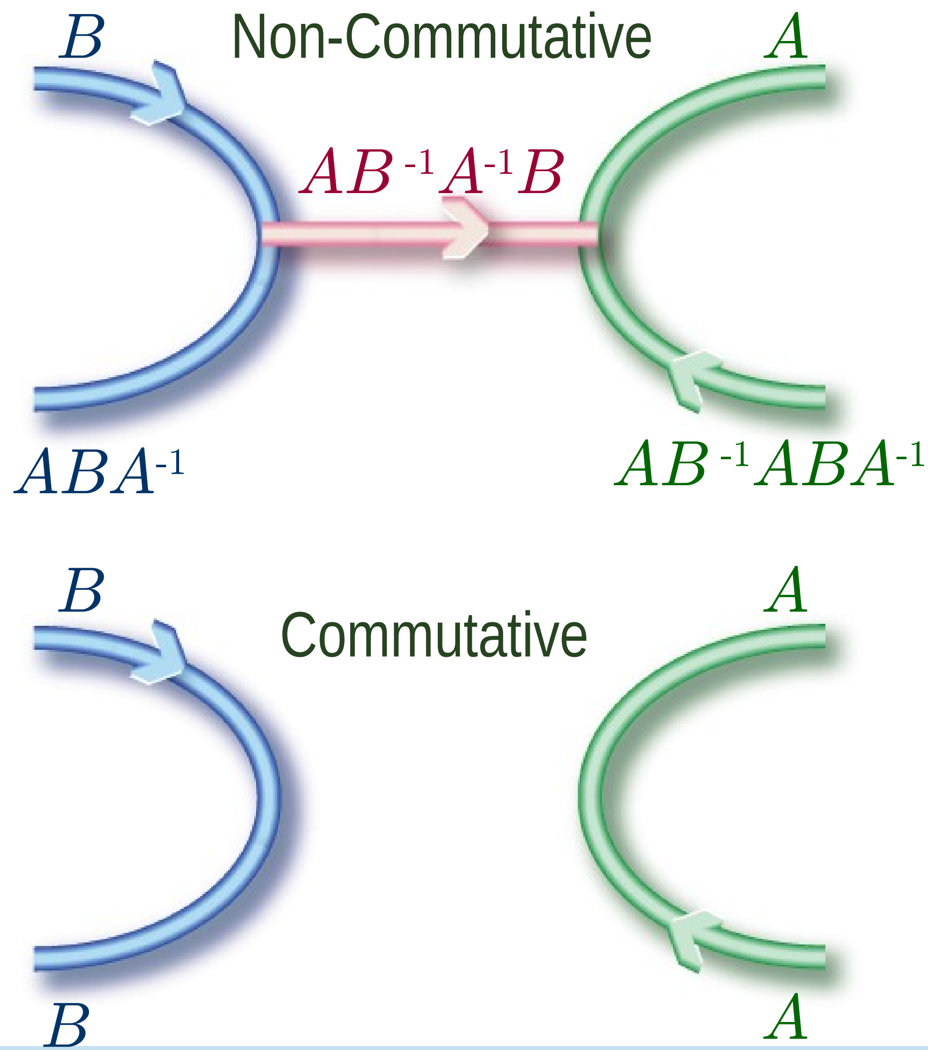


Non-Commutative Charge



Linked Rings





Characteristic Form of Wave Function

1/2 – spin vortex

$$\frac{\hat{S}}{2}\begin{pmatrix} i\exp[i\theta] \\ 0 \\ \sqrt{2} \\ 0 \\ i\exp[-i\theta] \end{pmatrix}$$

1/3 vortex

$$\frac{\hat{S}}{\sqrt{3}}\begin{pmatrix} \exp[i\theta] \\ 0 \\ 0 \\ \sqrt{2} \\ 0 \end{pmatrix}$$

$$\hat{S} = e^{i\varphi} e^{-i\hat{\boldsymbol{F}}\cdot\boldsymbol{\alpha}}$$

Vortex	Mass circulation (h/m)	Spin circulation (h/m)	Core
1/2 – spin	0	$1/2$	Nematic
1/3	$1/3$	$1/3$	Ferromagnetic

Homotopy Group of Cyclic State

Order-parameter manifold

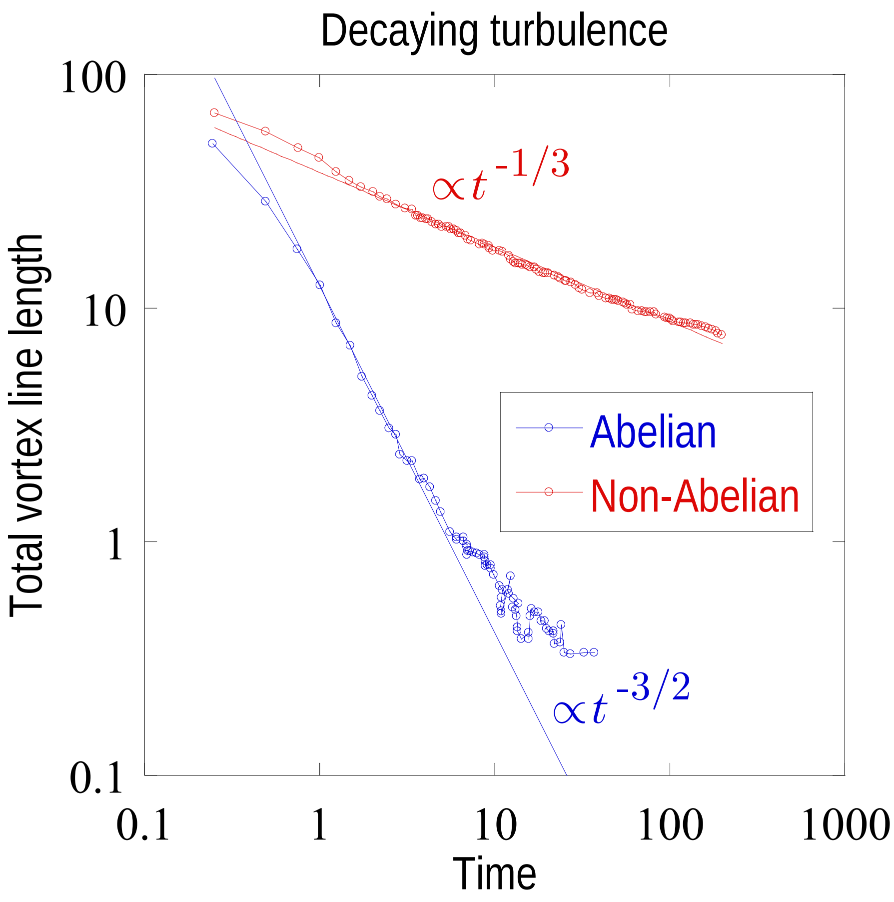
$$\frac{G}{H} \simeq \frac{U(1)_{\mathrm{G}} \times SO(3)_{\mathrm{S}}}{T_{\mathrm{S+G}}} \simeq \frac{U(1)_{\mathrm{G}} \times SU(2)_{\mathrm{S}}}{T^*_{\mathrm{S+G}}}$$

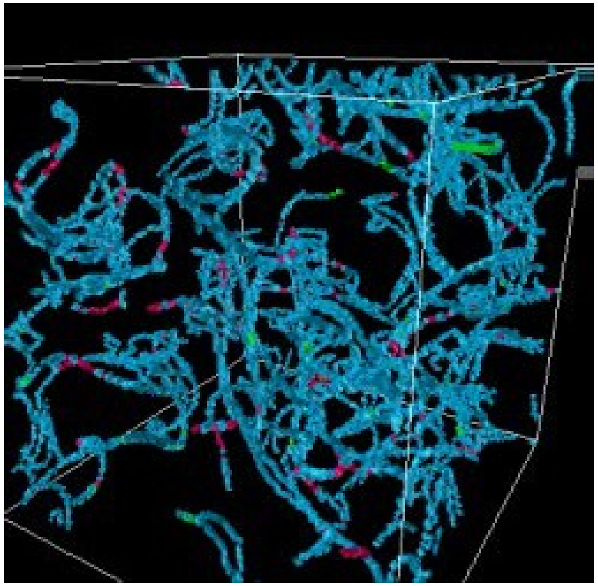
T^* : binary tetrahedral group

First homotopy group (vortex)

$$\pi_1\left(\frac{G}{H}\right) \simeq (\mathbb{Z} \times T^*)_{\mathrm{S+G}}$$

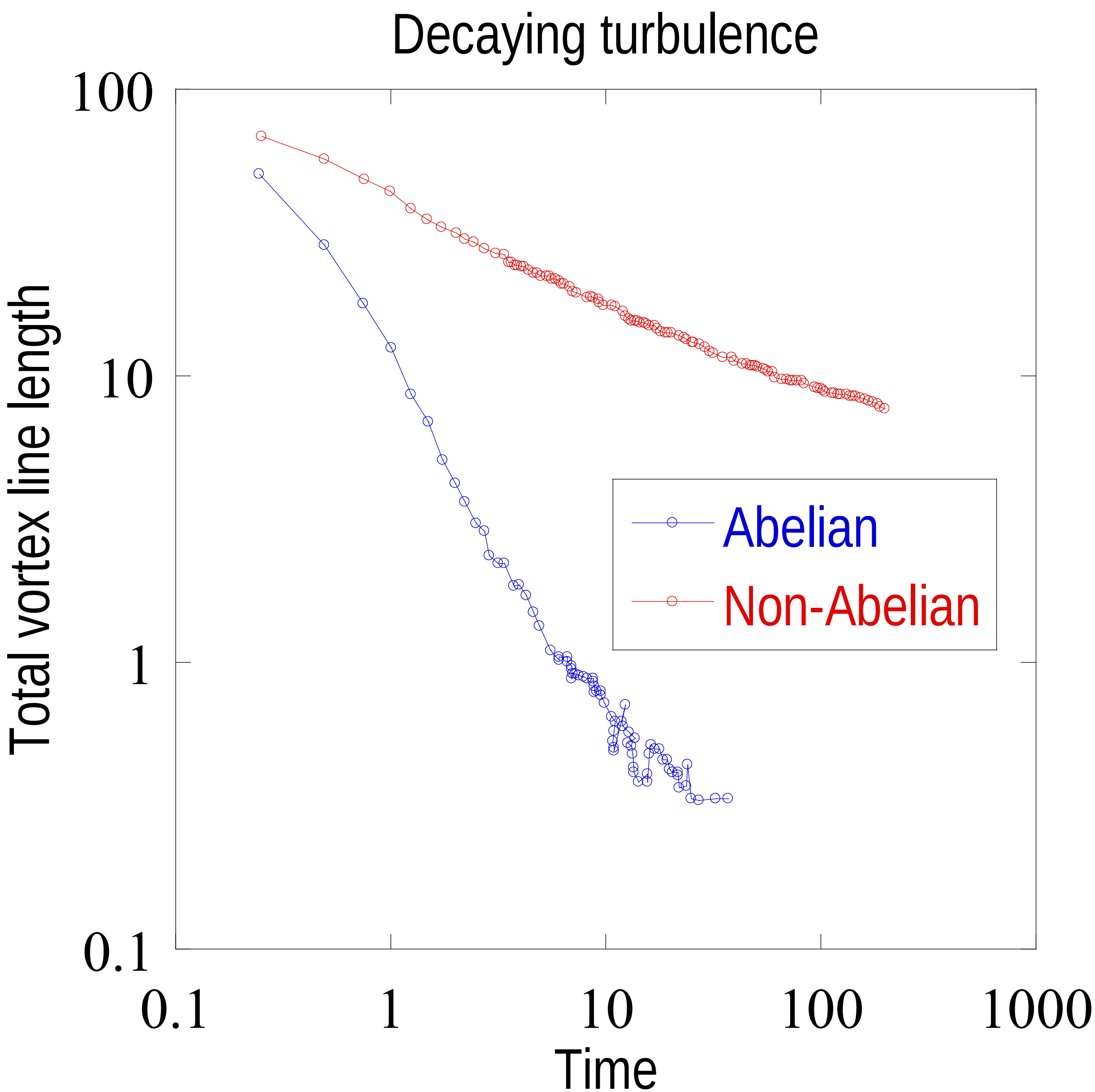
Non-Abelian Quantum Turbulence

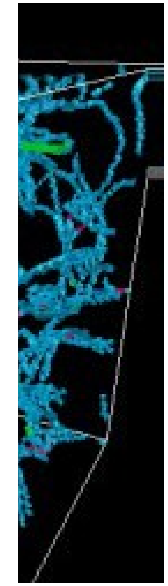




Turbulence with non-Abelian vortices does not seem to have classical analog.

Non… …nce





…n-Abelian
…seem to
…log.