Nonrelativistic Nambu-Goldstone Modes Associated with Spontaneously Broken Space-Time and Internal Symmetry

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- Nonrelativistic Nambu-Goldstone(NG) mode
- $\mathbb{CP}^1$ model
- NG modes associated with domain wall
- NG modes associated with baby skyrmion string

Feb. 3, 2015, “Topological soliton”

Nonrelativistic Nambu-Goldstone mode

Nambu-Goldstone modes and broken symmetries

Watanabe-Brauner relation

H. Watanabe and T. Brauner, Phys. Rev. D **84**, 125013 (2011)
H. Watanabe and H. Murayama, Phys. Rev. Lett. **108**, 251602 (2012)

$$N_{\mathrm{BG}} - N_{\mathrm{NG}} = \frac{1}{2}\mathrm{rank}\rho \qquad \rho_{ij} = \lim_{V\to\infty}\frac{1}{V}\int dV\,\langle[g_i, g_j]\rangle$$

N_{BG}: Number of broken symmetry generators
N_{NG}: Number of Nambu-Goldstone modes
g_i : Broken symmetry generators

$N_{\mathrm{BG}} > N_{\mathrm{NG}}$ (Coupling of two NG modes to one NG mode) can occurs when two generators do not commute

Nambu-Goldstone modes and broken symmetries

$$N_{\mathrm{BG}} - N_{\mathrm{NG}} = \frac{1}{2}\mathrm{rank}\rho \qquad \rho_{ij} = \lim_{V\to\infty}\frac{1}{V}\int dV\ \langle [g_i, g_j] \rangle$$

Nonlinear $O(3)$ sigma model & Heisenberg Ferromagnet

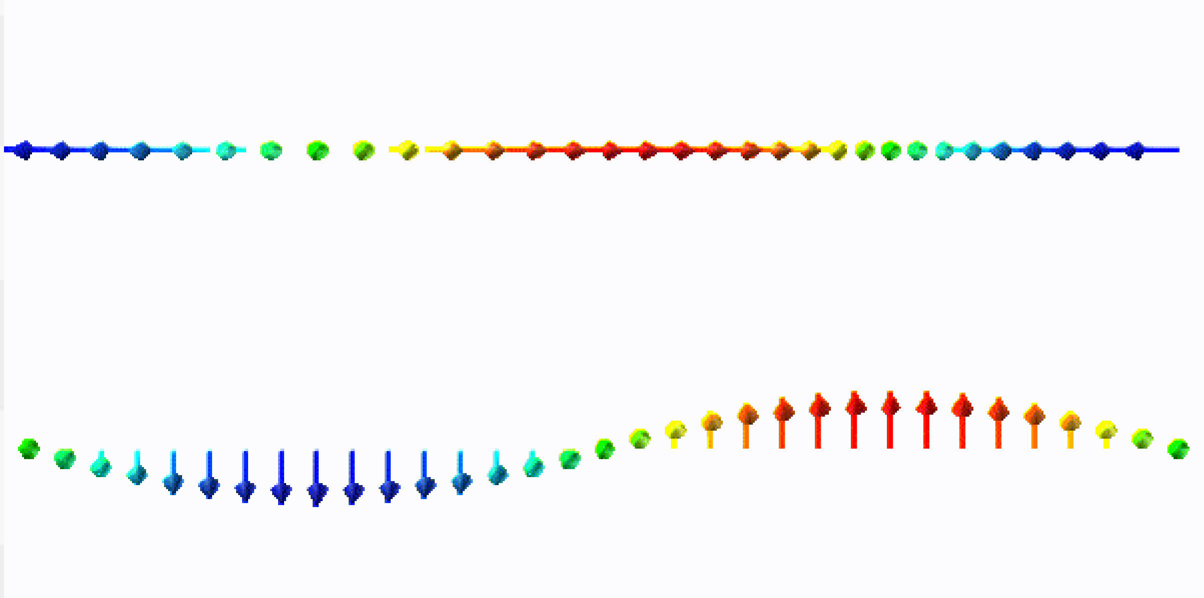
$$\mathcal{L}_{\mathrm{rel}} = \dot{\boldsymbol{n}}^2 - |\nabla \boldsymbol{n}|^2 \qquad \mathcal{L}_{\mathrm{nrel}} = \frac{2(\dot{n}_1 n_2 - n_1 \dot{n}_2)}{1+n_3} - |\nabla \boldsymbol{n}|^2$$

Symmetry breaking for ground state $\boldsymbol{n} = (0,0,1)$: $O(3) \to O(2)$

Symmetry generators : $\sigma_x,\ \sigma_y \ \to\ [\sigma_x, \sigma_y] = i\sigma_z$

$$N_{\mathrm{BG}} - N_{\mathrm{NG}} = \begin{cases} 0 \text{ for } \mathcal{L}_{\mathrm{rel}} \\ 1 \text{ for } \mathcal{L}_{\mathrm{nrel}} \end{cases}$$

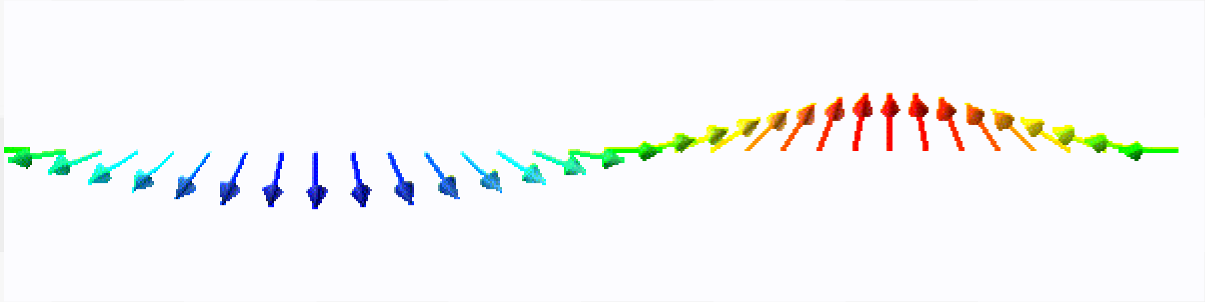
Nambu-Goldstone modes and broken symmetries



$$\mathcal{L}_{\rm rel} = \dot{\boldsymbol{n}}^2 - |\nabla \boldsymbol{n}|^2$$

: linear dispersion

$$\mathcal{L}_{\rm nrel} = \frac{2(\dot{n}_1 n_2 - n_1 \dot{n}_2)}{1+n_3} - |\nabla \boldsymbol{n}|^2$$: quadratic dispersion



$\mathbb{CP}^1$ model

$\mathbb{CP}^1$ model

$U(1)$ gauge theory

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{e^2}(\partial_\mu\Sigma)^2 + |D_\mu\phi|^2 - \frac{e^2}{2}(\phi^\dagger\phi - 1)^2 - \phi^\dagger(\Sigma\hat{1} - M)^2\phi$$

$\phi = (\phi_1, \phi_2)^T$: two charged complex scalar field

Σ : real scalar field

$M = \mathrm{diag}(m_1, m_2)$: mass

$\mathbb{CP}^1$ model

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{e^2}(\partial_\mu\Sigma)^2 + |D_\mu\phi|^2 - \frac{e^2}{2}(\phi^\dagger\phi - 1)^2 - \phi^\dagger(\Sigma\hat{1} - M)^2\phi$$

$$\Downarrow \quad \phi \to e^{-i\mu t}\phi$$

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{e^2}(\partial_\mu\Sigma)^2 + i\mu\{\phi^\dagger(D_t\phi) - (D_t\phi)^\dagger\phi\} + |D_\mu\phi|^2 - \frac{e^2}{2}(\phi^\dagger\phi - 1)^2 - \phi^\dagger(\Sigma\hat{1} - M)^2\phi + \mu^2c^2$$

$\phi = (\phi_1, \phi_2)^T$: two charged complex scalar field

Σ : real scalar field

$M = \mathrm{diag}(m_1, m_2)$: mass

$\mathbb{CP}^1$ model

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{e^2}(\partial_\mu\Sigma)^2 + i\mu\{\phi^\dagger(D_t\phi) - (D_t\phi)^\dagger\phi\}$$

$$+ |D_\mu\phi|^2 - \frac{e^2}{2}(\phi^\dagger\phi - 1)^2 - \phi^\dagger(\Sigma\hat{1} - M)^2\phi + \mu^2 c^2$$

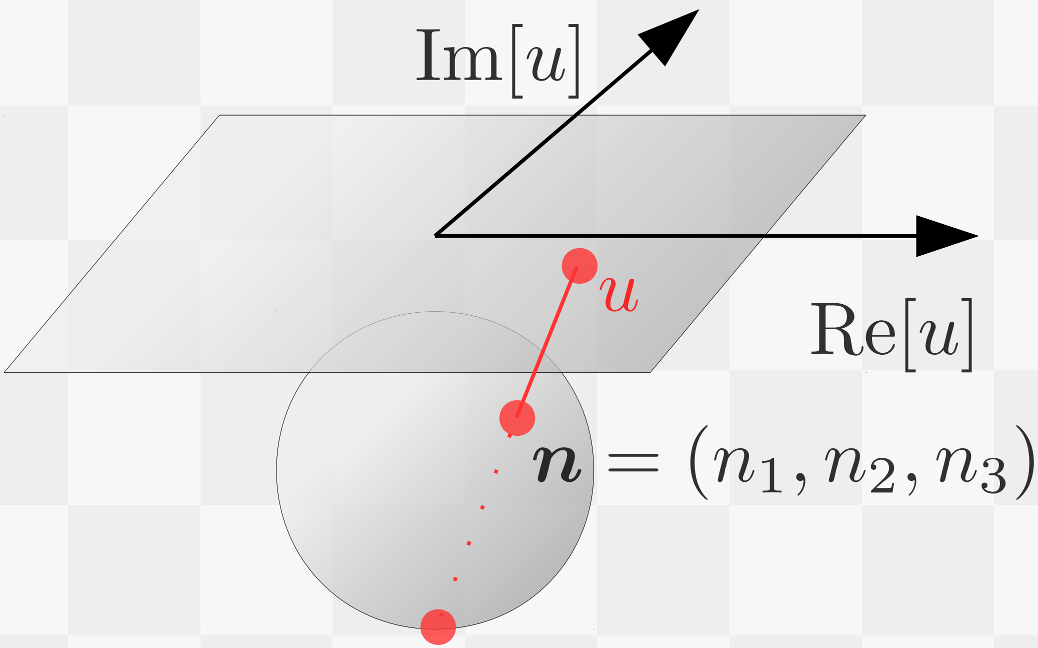
$e \to \infty, \quad \phi = (1, u)^T/\sqrt{1+|u|^2}$: $\mathbb{CP}^1$ form

$$\mathcal{L} - \mu^2 c^2 = \frac{|\dot{u}|^2}{c^2(1+|u|^2)^2} - \frac{|\nabla u|^2}{(1+|u|^2)^2} + \frac{i\mu(u^*\dot{u} - \dot{u}^*u)}{1+|u|^2}$$

$$- \frac{m^2|u|^2}{(1+|u|^2)^2} \qquad m_1 - m_2 \equiv m$$

$\mathbb{CP}^1$ model

$\mathbb{CP}^1$: Projection from south pole on S^2 to complex plane



$$u = \frac{n_1 + i n_2}{1 + n_3}$$

$$n_1 = \pm 1 \;\to\; u = \pm 1$$

$$n_2 = \pm 1 \;\to\; u = \pm i$$

$$n_3 = 1 \;\to\; u = 0$$

$$n_3 = -1 \;\to\; u = \infty$$

$\mathbb{CP}^1$ model

$$\mathcal{L} - \mu^2 c^2 = \frac{|\dot{u}|^2}{c^2(1+|u|^2)^2} - \frac{|\nabla u|^2}{(1+|u|^2)^2} + \frac{i\mu(u^*\dot{u} - \dot{u}^* u)}{1+|u|^2} - \frac{m^2|u|^2}{(1+|u|^2)^2}$$

ultrarelativistic limit : $\mu \to 0$ massive $O(3)$ sigma model

$$\mathcal{L} = \frac{|\dot{u}|^2}{c^2(1+|u|^2)^2} - \frac{|\nabla u|^2}{(1+|u|^2)^2} - \frac{m^2|u|^2}{(1+|u|^2)^2}$$

nonrelativistic limit : $c \to \infty$ Heisenberg ferromagnet with one easy axis (Ising ferromagnet)

$$\mathcal{L} - \mu^2 c^2 = -\frac{|\nabla u|^2}{(1+|u|^2)^2} + \frac{i\mu(u^*\dot{u} - \dot{u}^* u)}{1+|u|^2} - \frac{m^2|u|^2}{(1+|u|^2)^2}$$

Internal symmetry of $\mathbb{CP}^1$ model

$$\mathcal{L} - \mu^2 c^2 = \frac{|\dot{u}|^2}{c^2(1+|u|^2)^2} - \frac{|\nabla u|^2}{(1+|u|^2)^2} + \frac{i\mu(u^*\dot{u} - \dot{u}^* u)}{1+|u|^2}$$

$$- \frac{m^2|u|^2}{(1+|u|^2)^2}$$

$$\left.\begin{array}{c} u \to u^{i\alpha} \;\; : \;\; SO(2) \\ \& \\ u \to 1/u^* \;\; : \;\; \mathbb{Z}_2 \end{array}\right\} SO(2) \rtimes \mathbb{Z}_2 \quad \text{for } m \neq 0$$

$$u \to \text{arbitrary } u \;\; : \;\; \mathbb{CP}^1 \simeq S^2 \qquad \text{for } m = 0$$

Spacetime symmetry of $\mathbb{CP}^1$ model

$$\mathcal{L} - \mu^2 c^2 = \frac{|\dot{u}|^2}{c^2(1+|u|^2)^2} - \frac{|\nabla u|^2}{(1+|u|^2)^2} + \frac{i\mu(u^*\dot{u} - \dot{u}^*u)}{1+|u|^2} - \frac{m^2|u|^2}{(1+|u|^2)^2}$$

for $\mu \neq 0$: Lorentz symmetry with phase shift

$$t \to \gamma\left(t - \frac{\boldsymbol{v}\cdot\boldsymbol{x}}{c^2}\right) \qquad \boldsymbol{x} \to \gamma(\boldsymbol{x} - \boldsymbol{v}t) \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

$$u \to u^{i\mu\mathcal{S}} \quad \dot{\mathcal{S}} = -(1-\gamma)c^2(1+|u|^2) \quad \nabla\mathcal{S} = -\gamma(1+|u|^2)$$

Spacetime symmetry of $\mathbb{CP}^1$ model

$$\mathcal{L} = \frac{|\dot{u}|^2}{c^2(1+|u|^2)^2} - \frac{|\nabla u|^2}{(1+|u|^2)^2} - \frac{m^2|u|^2}{(1+|u|^2)^2}$$

for $\mu = 0$: simple Lorentz symmetry

$$t \to \gamma\left(t - \frac{\boldsymbol{v}\cdot\boldsymbol{x}}{c^2}\right) \qquad \boldsymbol{x} \to \gamma(\boldsymbol{x} - \boldsymbol{v}t) \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Spacetime symmetry of $\mathbb{CP}^1$ model

$$\mathcal{L} - \mu^2 c^2 = -\frac{|\nabla u|^2}{(1+|u|^2)^2} + \frac{i\mu(u^* \dot{u} - \dot{u}^* u)}{1+|u|^2} - \frac{m^2|u|^2}{(1+|u|^2)^2}$$

for $c \to \infty$: Galilean symmetry with phase shift

$$t \to t \qquad \boldsymbol{x} \to \boldsymbol{x} - \boldsymbol{v}t$$

$$u \to u^{i\mu\mathcal{S}} \quad \dot{\mathcal{S}} = -v^2(1+|u|^2)/2 \quad \nabla\mathcal{S} = -(1+|u|^2)$$

Ground-state symmetry

$$\mathcal{L} - \mu^2 c^2 = \frac{|\dot{u}|^2}{c^2(1+|u|^2)^2} - \frac{|\nabla u|^2}{(1+|u|^2)^2} + \frac{i\mu(u^*\dot{u} - \dot{u}^* u)}{1+|u|^2} - \frac{m^2|u|^2}{(1+|u|^2)^2}$$

For internal symmetry

	state	G	H	G/H
$m \neq 0$	$u = 0,\ \infty$	$SO(2) \rtimes \mathbb{Z}_2$	$SO(2)$	$\mathbb{Z}_2$
$m = 0$	arbitrary u	$\mathbb{C}^2$	$\mathbb{C} - 0$	$\mathbb{CP}^1$

Ground-state symmetry

$$\mathcal{L} - \mu^2 c^2 = \frac{|\dot{u}|^2}{c^2(1+|u|^2)^2} - \frac{|\nabla u|^2}{(1+|u|^2)^2} + \frac{i\mu(u^*\dot{u} - \dot{u}^*u)}{1+|u|^2} - \frac{m^2|u|^2}{(1+|u|^2)^2}$$

For spacetime symmetry

	G	H	G/H
$\mu \neq 0$	Lorentz	translation & Euclid	Lorentz boost
$\mu = 0$	Lorentz	Lorentz	1
$c \to \infty$	Galilean	translation & Euclid	Galilean boost

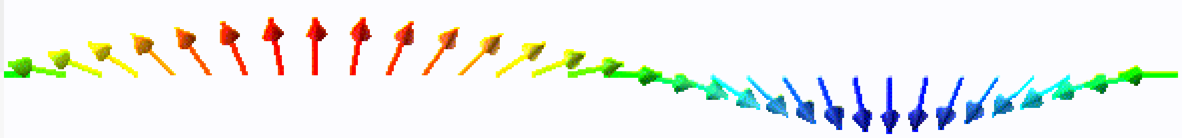
Nambu-Goldstone mode for $m = 0$

$$\mathcal{L} - \mu^2 c^2 = \frac{|\dot{u}|^2}{c^2(1+|u|^2)^2} - \frac{|\nabla u|^2}{(1+|u|^2)^2} + \frac{i\mu(u^*\dot{u} - \dot{u}^* u)}{1+|u|^2}$$

$u = u_0 + \delta u \quad u_0 = 0$

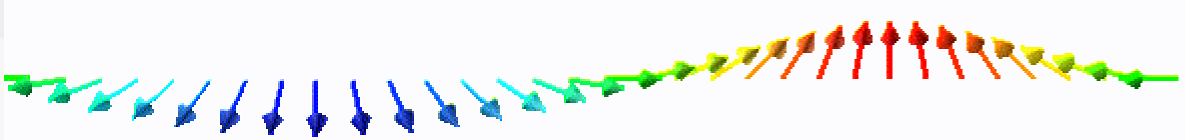
$$\omega_\pm = c\sqrt{k^2 + c^2\mu^2} \pm c^2\mu = \frac{k^2}{2\mu} + c^2\mu \pm c^2\mu + O(k^3)$$

ω_+ : gapful mode



ω_- : NG mode

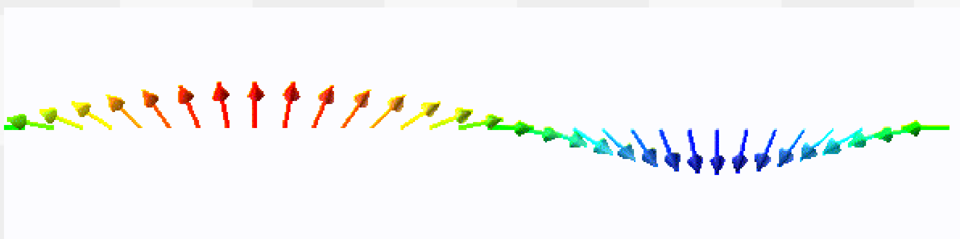
quadratic dispersion



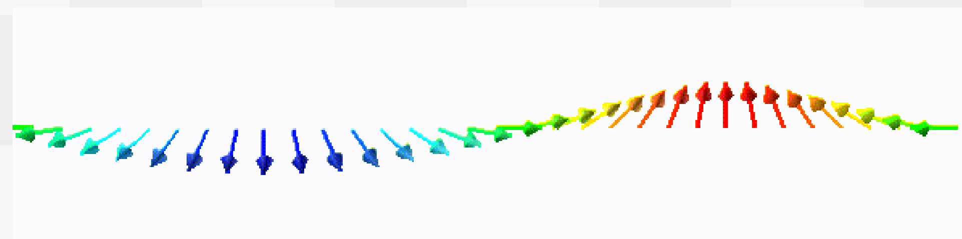
Nambu-Goldstone mode for $m = 0$

ultrarelativistic limit : $\omega_{\pm} = ck$ (linear dispersion)

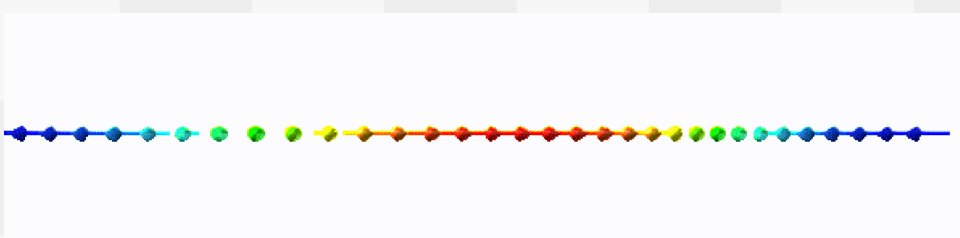
ω_{+}



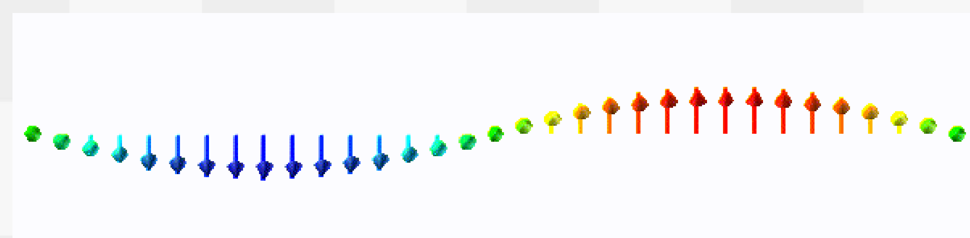
ω_{-}



$\omega_{\parallel}$



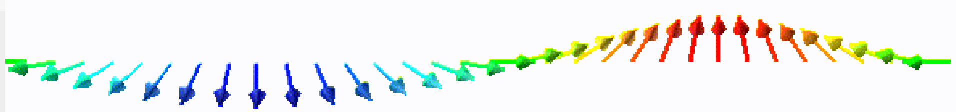
$\omega_{\perp}$



Nambu-Goldstone mode for $m = 0$

nonrelativistic limit : $\omega_+ \to \infty \quad \omega_- = \dfrac{k^2}{2\mu} + O(k^3)$

ω_- : NG mode



NG modes associated with domain wall

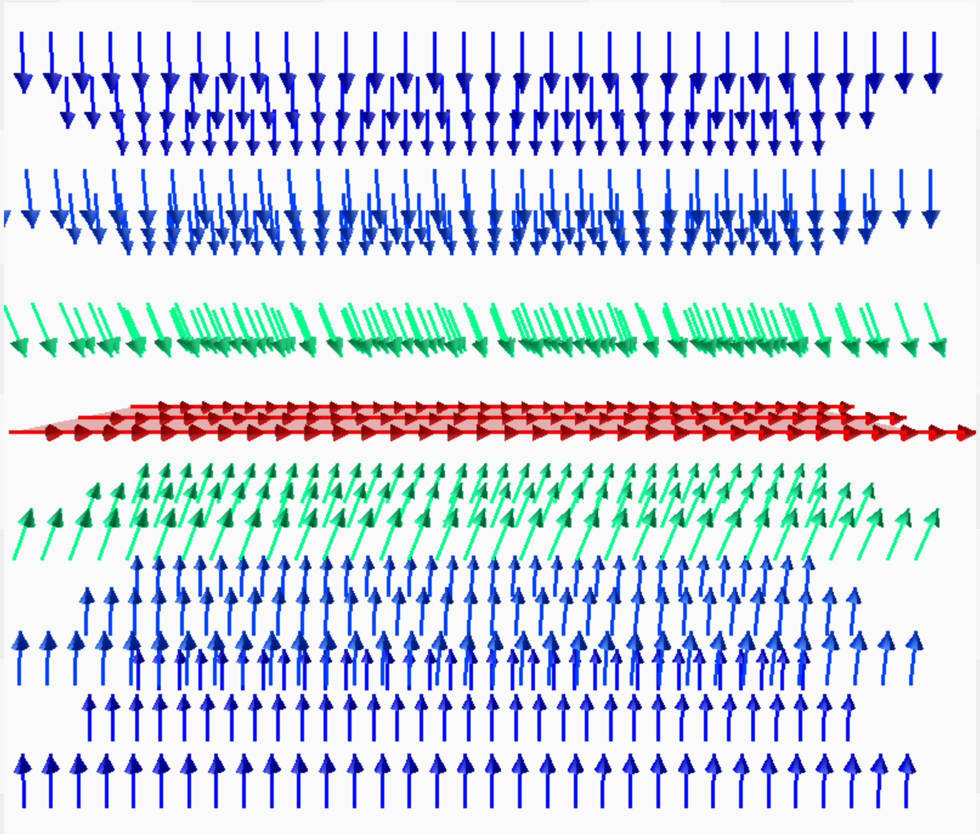
Domain wall solution

$$\mathcal{L} - \mu^2 c^2 = \frac{|\dot{u}|^2}{c^2(1+|u|^2)^2} - \frac{|\nabla u|^2}{(1+|u|^2)^2} + \frac{i\mu(u^*\dot{u} - \dot{u}^* u)}{1+|u|^2} - \frac{m^2|u|^2}{(1+|u|^2)^2}$$

ground state : $u_0 = 0,\ \infty$

domain-wall solution

$u_{\mathrm{DW}} = e^{im(z-Z)+i\alpha}$



Z : translational moduli

(translational symmetry)

α : phase moduli

(internal symmetry)

Low energy dynamics of domain wall

Linear response theory

domain-wall solution

$u_{\mathrm{DW}} = e^{im(z-Z)+i\alpha}$ ⇒ $u = u_{\mathrm{DW}} + \delta u$

$$\delta u = a_+(z)e^{i\boldsymbol{k}\cdot\boldsymbol{r}-\omega t} + a_-^*(z)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}-\omega t}$$

⇓

Bogoliubov-de Gennes equation

$$(\omega^2/c^2 \pm 2\mu\omega)a_\pm$$
$$= \left\{(\boldsymbol{k}_\perp^2 - \partial_z^2) + \frac{4me^{2mz}\partial_z - m^2(3e^{2mz}-1)}{1+e^{2mz}}\right\}a_\pm + O(a_\pm^2)$$

Low energy dynamics of domain wall

$$(\omega^2/c^2 \pm 2\mu\omega)a_\pm$$

$$= \left\{ (\boldsymbol{k}_\perp^2 - \partial_z^2) + \frac{4me^{2mz}\partial_z - m^2(3e^{2mz} - 1)}{1 + e^{2mz}} \right\} a_\pm$$

$$\omega_{\mathrm{H}} = \sqrt{\mu^2 c^4 + c^2 k^2} + \mu c^2$$

$$= \frac{k^2}{2\mu} + 2\mu c^2 + O(k^3) \ : \ \text{gapful mode}$$

$$\omega_{\mathrm{NG}} = \sqrt{\mu^2 c^4 + c^2 k^2}$$

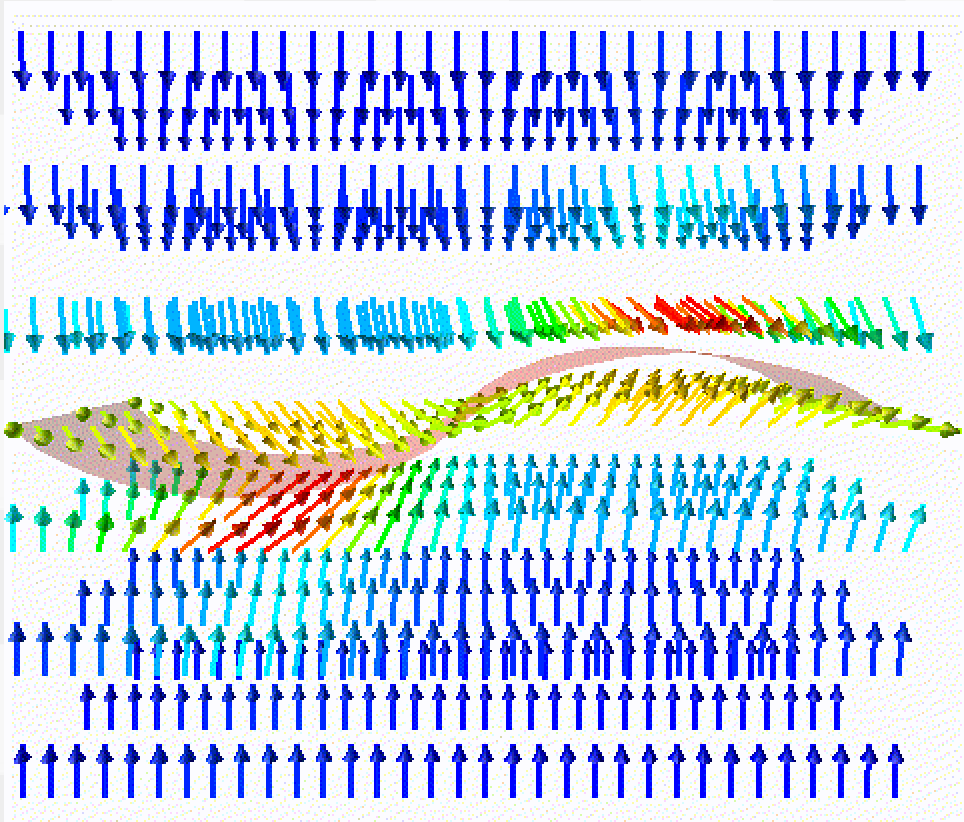
$$= \frac{k^2}{2\mu} + O(k^3) \ : \ \text{Nambu-Goldstone mode}$$

(quadratic dispersion)

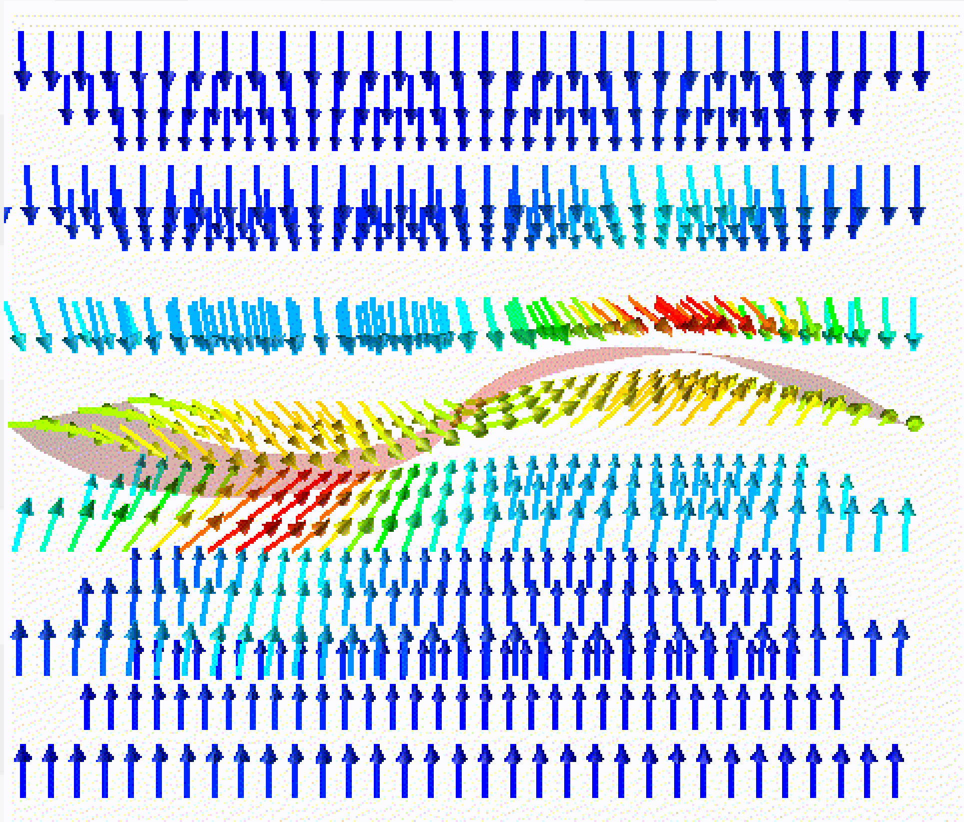
Nambu-Goldstone and gapful (Higgs) modes

$$\omega_{\rm NG} = \frac{k^2}{2\mu} + O(k^3)$$

$$\omega_{\rm H} = \frac{k^2}{2\mu} + 2\mu c^2 + O(k^3)$$



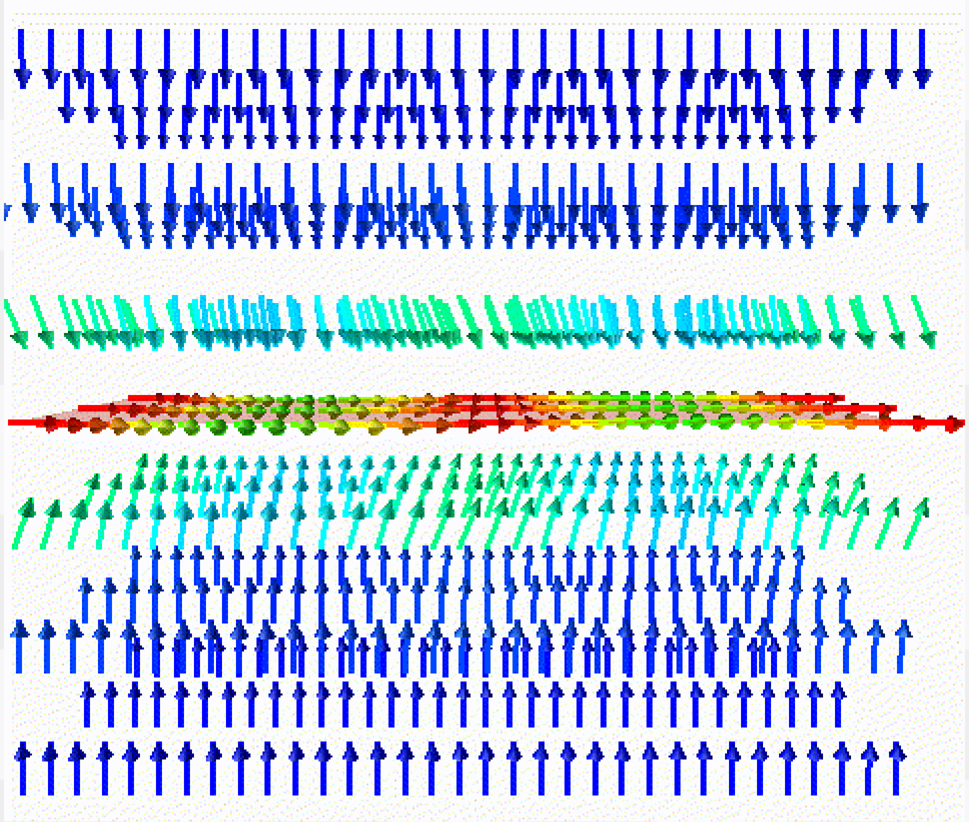
Coupling of translational and phase moduli



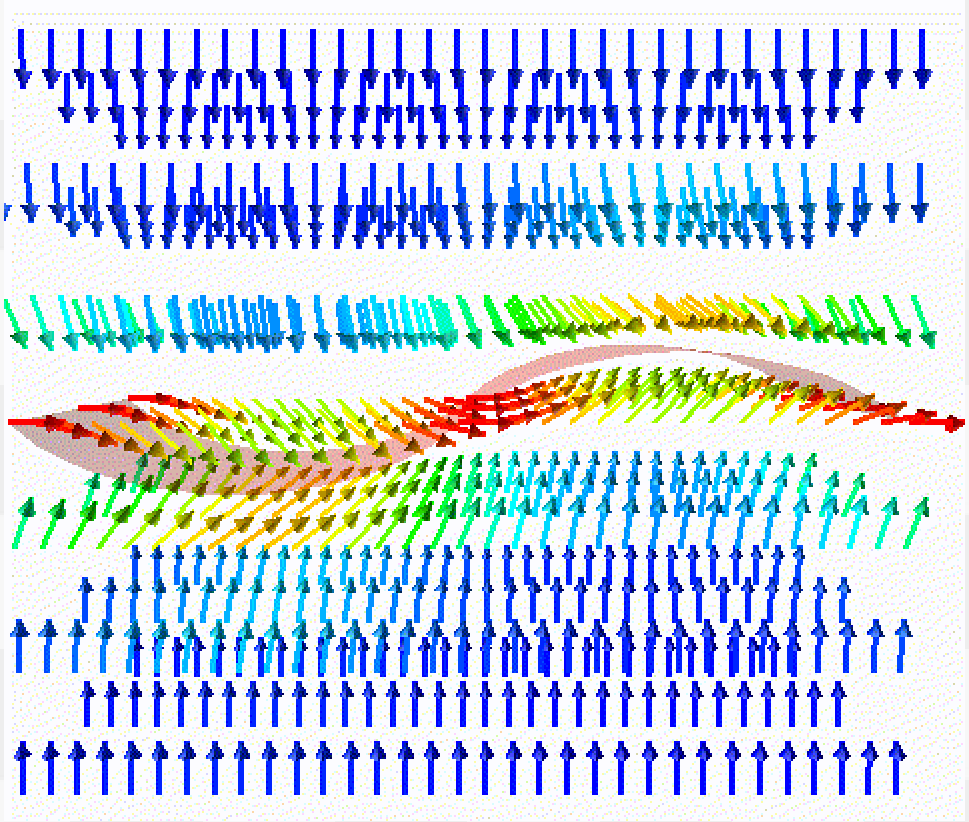
Inverse coupling of moduli to those of NG mode

Ultrarelativistic limit ($\mu \to 0$)

$$\omega_{\mathrm{H}},\ \omega_{\mathrm{NG}} \to ck$$



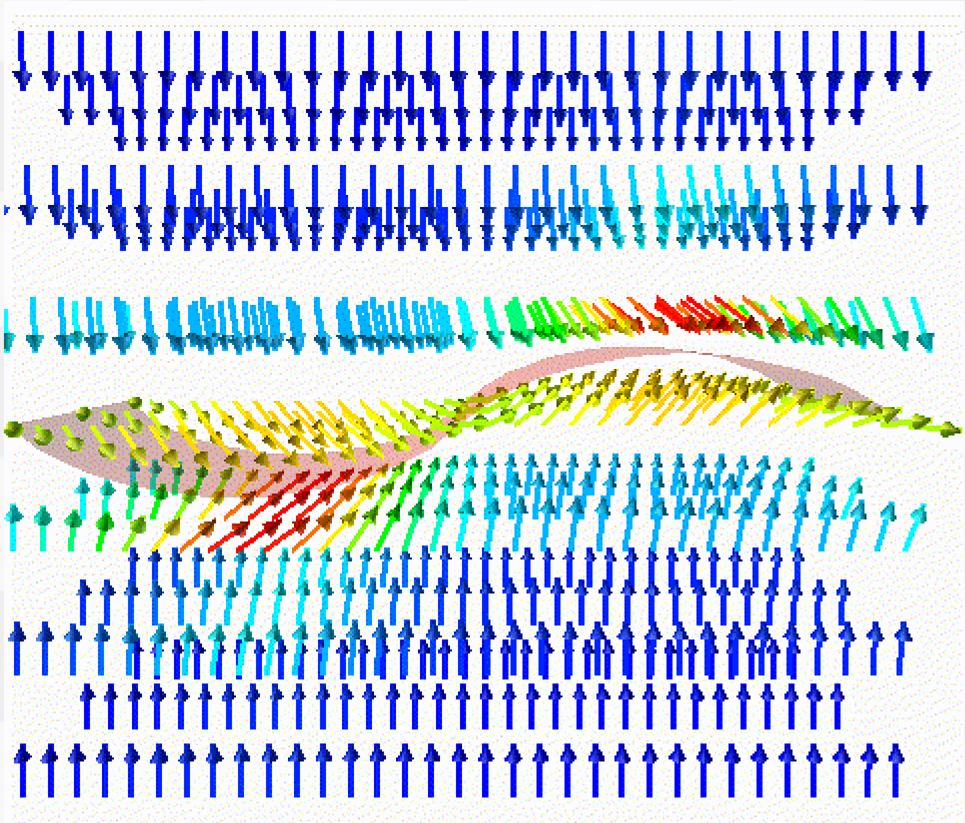
phase moduli



translational moduli

Nonrelativistic limit ($c \to \infty$)

$$\omega_{\mathrm{NG}} = \frac{k^2}{2\mu} + O(k^3) \qquad \omega_{\mathrm{H}} \to \infty$$



Commutation relation between symmetric generators

conjugate variable of u : $v = \frac{\partial \mathcal{L}}{\partial \dot{u}} = \frac{\dot{u}^*}{c^2(1+|u|^2)^2} + \frac{i\mu u^*}{1+|u|^2}$

generator for translation : $P_z = \int dz\, (\partial_z u) v$

generator for phase shift : $\Theta = \int dz\, (iu) v$

$$[P_z, \Theta] = \mu \left[\frac{|u|^2}{1+|u|^2} \right]_{z=-\infty}^{z=+\infty} \equiv \mu W$$

$W = \left[\frac{|u|^2}{1+|u|^2} \right]_{z=-\infty}^{z=+\infty} = 1$: domain wall charge for $\pi_0(\mathbb{Z}_2) \simeq \mathbb{Z}_2$

NG modes associated with baby skyrmion string

Baby skyrmion solution

$$\mathcal{L} - \mu^2 c^2 = \frac{|\dot{u}|^2}{c^2(1+|u|^2)^2} - \frac{|\nabla u|^2}{(1+|u|^2)^2} + \frac{i\mu(u^*\dot{u} - \dot{u}^*u)}{1+|u|^2}$$

ground state : $u = \mathrm{const}$

skyrmion solution

$$u_{\mathrm{S}} = \frac{\exp\left\{i\left(\tan^{-1}\frac{y-Y}{x-X} + \alpha\right)\right\}\sqrt{(x-X)^2+(y-Y)^2}}{R_0 + R}$$

X, Y : translational moduli (translational symmetry)

α : phase moduli (internal symmetry)

R : dilatation moduli (dilatation symmetry)

Baby skyrmion solution

skyrmion solution

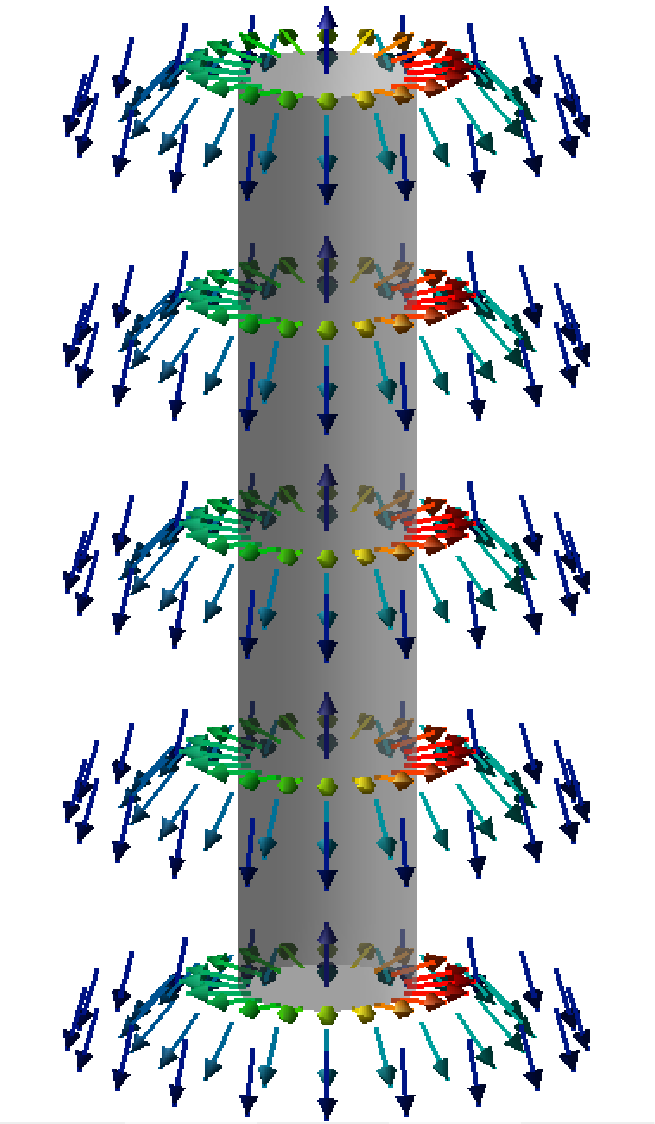
$$u_{\rm S} = \exp\left\{ i \left(\tan^{-1} \frac{y - Y}{x - X} + \alpha \right) \right\}$$

$$\times \frac{\sqrt{(x - X)^2 + (y - Y)^2}}{R_0 + R}$$

X, Y : translational moduli

α : phase moduli

R : dilatation moduli



Dilatation moduli

Dilatation symmetry is not the symmetry of the action : $S = \int d^4x\, \mathcal{L}$

$\delta S = 0 \rightarrow$

$$\frac{(1+|u|^2)\ddot{u} - 2u^*\dot{u}^2}{c^2} - 2i\mu(1+|u|^2)\dot{u} = (1+|u|^2)\nabla^2 u - 2u^*(\nabla u)^2$$

stationary state

$(1+|u|^2)\nabla^2 u - 2u^*(\nabla u)^2 = 0$: invariant under dilatation $\boldsymbol{r} \rightarrow \kappa \boldsymbol{r}$

Dilatation symmetry is the symmetry of the stationary state of the dynamical equation

Low energy dynamics of domain wall

Linear response theory

$$u = u_{\mathrm{S}} + \delta u$$

$$\delta u = \sum_l \left\{ a_{+,l}(r) e^{i(kz + l\phi - \omega t)} + a^*_{-,l}(r) e^{-i(kz + l\phi - \omega t)} \right\}$$

Bogoliubov-de Gennes equation

$$(\omega^2/c^2 \pm 2\mu\omega) a_{\pm,l}$$
$$= \left\{ (k^2 - \partial_r^2 - (1/r)\partial_r + l^2/r^2) + \frac{4(r\partial_r \mp l)}{r^2 + R_0^2} \right\} a_{\pm,l} + O(a_{\pm,l}^2)$$

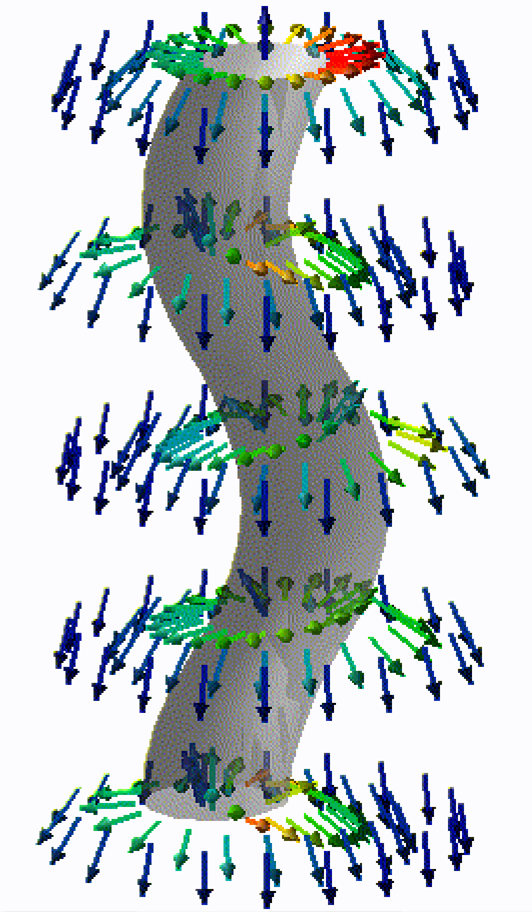
Low energy dynamics of domain wall

$$(\omega^2/c^2 \pm 2\mu\omega)a_{\pm,l}$$
$$= \left\{(k^2 - \partial_r^2 - (1/r)\partial_r + l^2/r^2) + \frac{4(r\partial_r \mp l)}{r^2 + R_0^2}\right\}a_{\pm,l} + O(a_{\pm,l}^2)$$

For arbitrary l

$$\omega_{\rm H} = \sqrt{\mu^2c^4 + c^2k^2} + \mu c^2$$
$$= \frac{k^2}{2\mu} + 2\mu c^2 + O(k^3) \ : \ \text{gapful mode}$$
$$\omega_{\rm L} = \sqrt{\mu^2c^4 + c^2k^2} - \mu c^2$$
$$= \frac{k^2}{2\mu} + O(k^3) \ : \ \text{gapless mode}$$
$$\omega_{\rm H} = \omega_{\rm L} = ck \ : \ \text{ultrarelativistic limit}$$

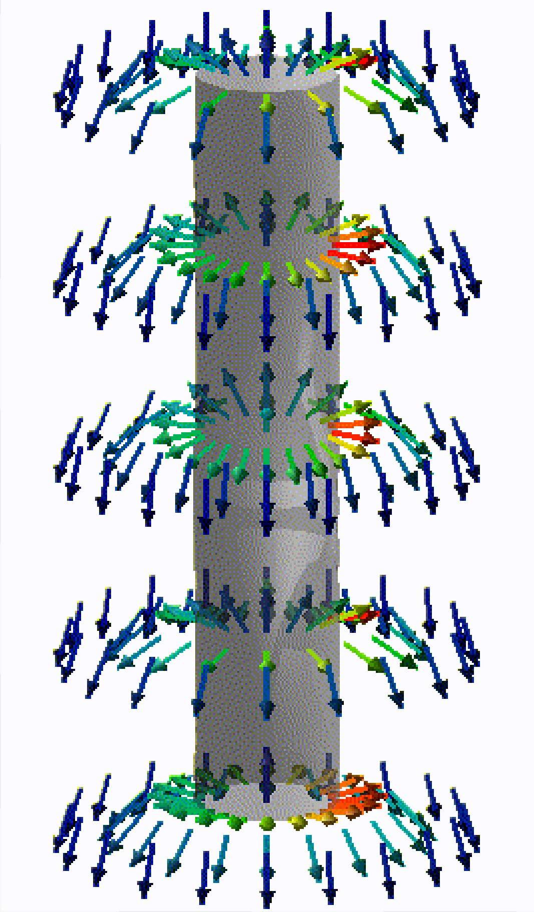
Ultrarelativistic limit



$$\omega_{\mathrm{H}} = \omega_{\mathrm{L}} = ck$$

For $l = 0$

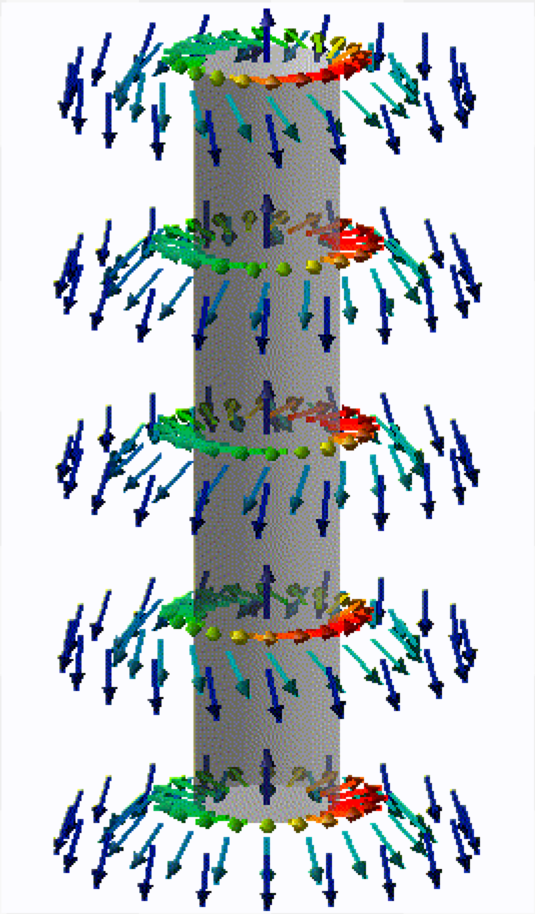
Two NG modes



Translational mode along x direction

Translational mode along y direction

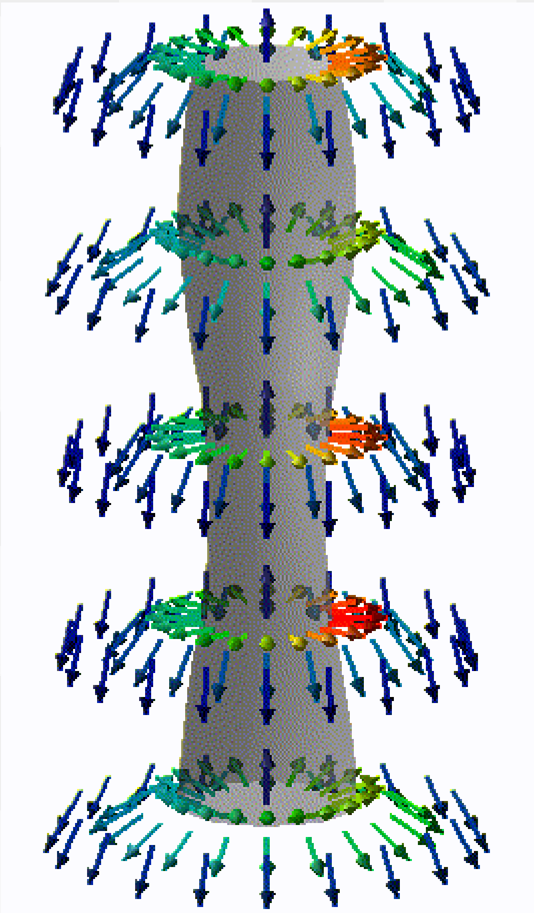
Ultrarelativistic limit



$$\omega_{\mathrm{H}} = \omega_{\mathrm{L}} = ck$$

For $l = 1$

One NG and
one QNG modes



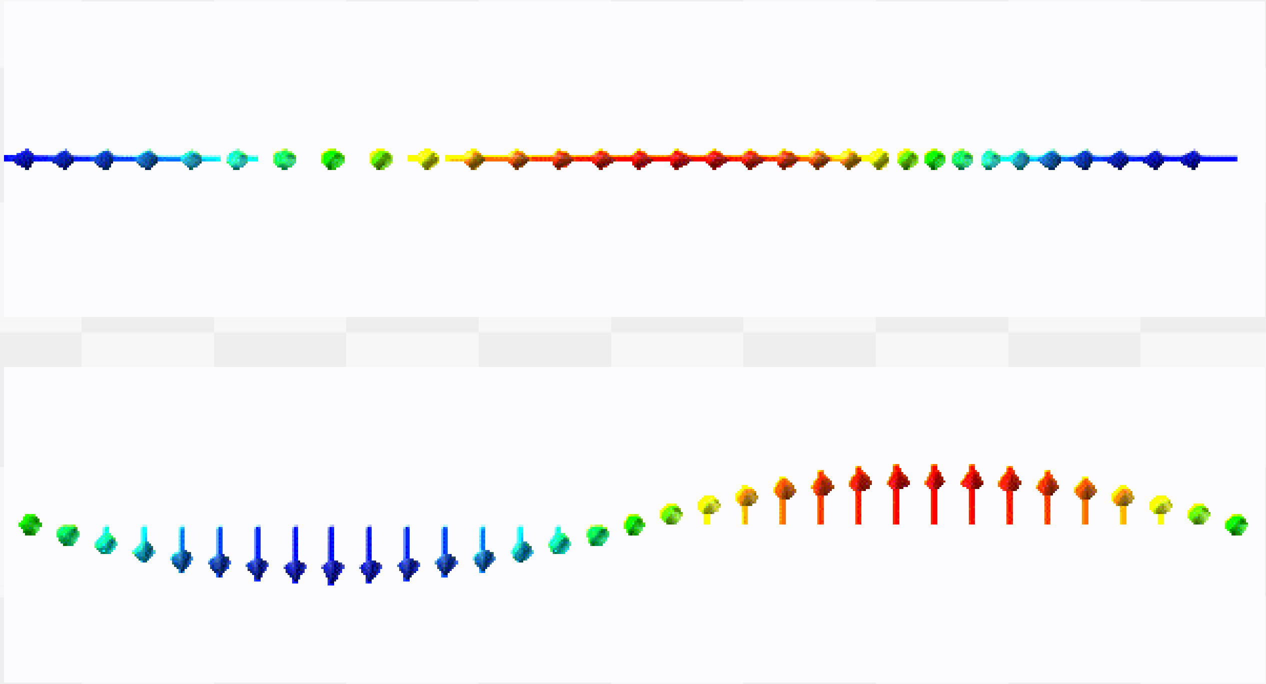
Phase mode : NG mode

Dilatation mode :
quasi NG mode

Ultrarelativistic limit

$$\omega_{\mathrm{H}} = \omega_{\mathrm{L}} = ck$$

For $l = 2$: bulk mode far from skyrmion

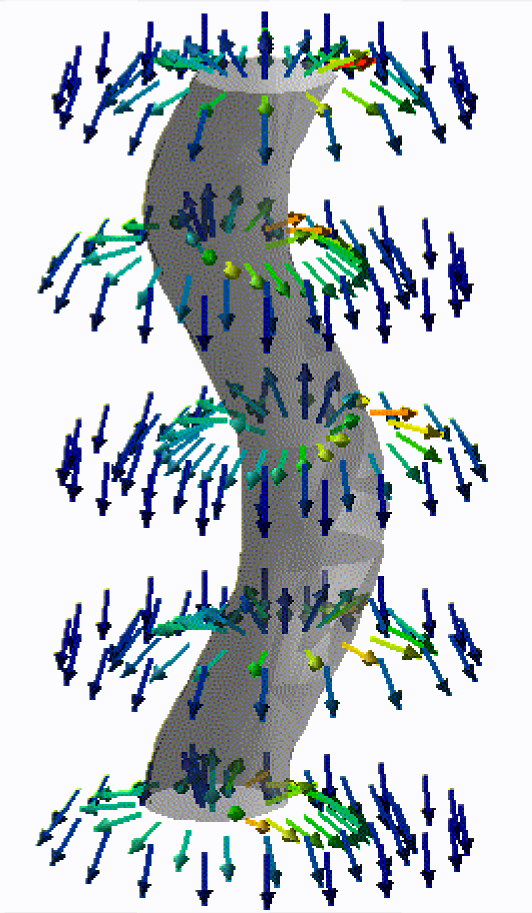


Relativistic mode

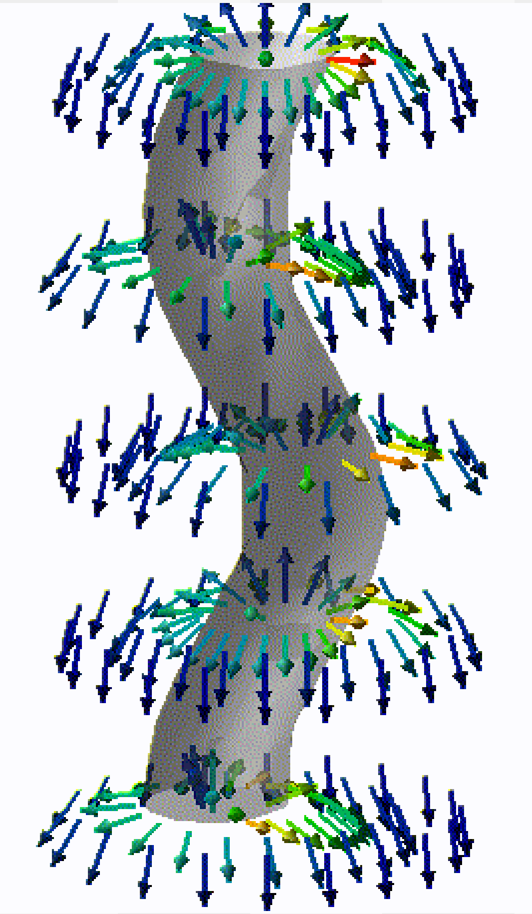
$$\omega_{\mathrm{L}} = \frac{k^2}{2\mu} + O(k^3)$$

$$\omega_{\mathrm{H}} = \frac{k^2}{2\mu} + 2\mu c^2 + O(k^3)$$

For $l = 0$



Coupling of two translational (NG) modes

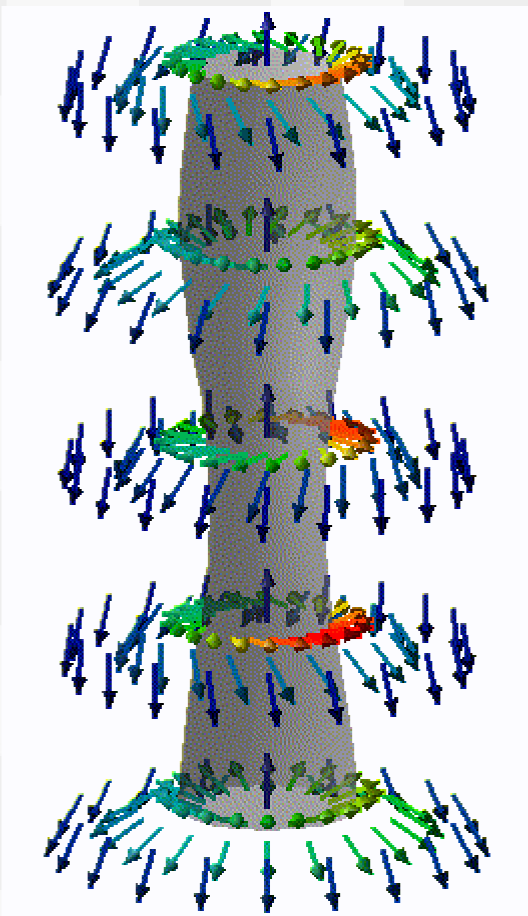


Relativistic mode

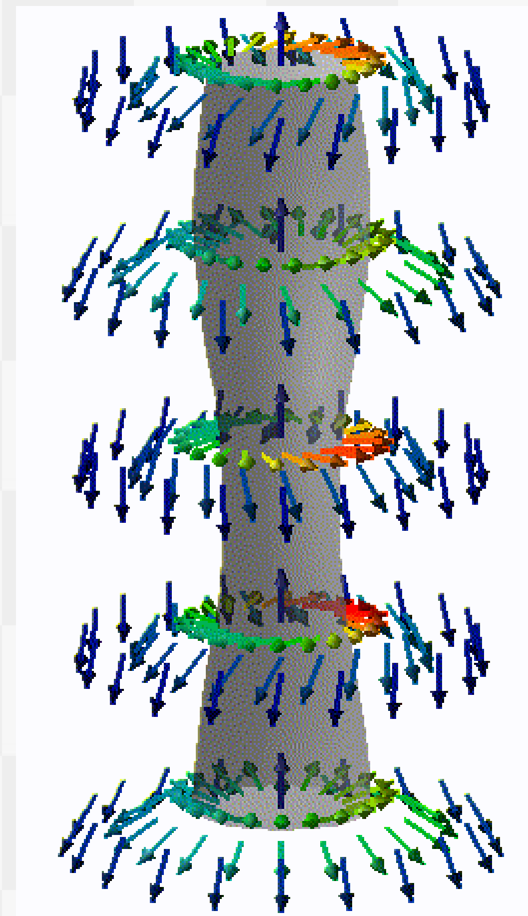
$$\omega_{\mathrm{L}} = \frac{k^2}{2\mu} + O(k^3)$$

$$\omega_{\mathrm{H}} = \frac{k^2}{2\mu} + 2\mu c^2 + O(k^3)$$

For $l = 1$



Coupling of phase (NG) and dilatation (QNG) modes



Commutation relation between symmetric generators

conjugate variable of u : $v = \dfrac{\partial \mathcal{L}}{\partial \dot{u}} = \dfrac{\dot{u}^*}{c^2(1+|u|^2)^2} + \dfrac{i\mu u^*}{1+|u|^2}$

generator for translation : $P_{x,y} = \int d^2x \, (\partial_{x,y} u) v$

generator for phase shift : $\Theta = \int d^2x \, (iu) v$

generator for dilatation : $D = \int d^2x \, (x\partial_x u + y\partial_y u) v$

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$$[P_x, P_y] = \mu \int d^2x \, \frac{(\partial_r u)(\partial_\phi v) - (\partial_\phi u)(\partial_r v)}{r} \equiv \mu B$$

$$B = \int d^2x \, \frac{(\partial_r u)(\partial_\phi v) - (\partial_\phi u)(\partial_r v)}{r} = \int d^2x \, b = 2\pi$$

: skyrmion charge for $\pi_2(\mathbb{CP}^1) \simeq \mathbb{Z}$

Commutation relation between symmetric generators

generator for translation : $P_{x,y} = \int d^2x \, (\partial_{x,y} u) v$

generator for phase shift : $\Theta = \int d^2x \, (iu) v$

generator for dilatation : $D = \int d^2x \, (x\partial_x u + y\partial_y u) v$

$[D, \Theta] = \mu \int d^2x \, r^2 \left(b + \frac{1}{r^2 + R_0^2} \right)$: not topological number

All other commutation relations vanish

Summary

- For systems with spontaneously broken Lorentz symmetry (or without Lorentz symmetry), two NG may couple to one NG with quadratic dispersion
- For domain wall case in massive $\mathbb{CP}^1$ model, translational NG mode (spacetime symmetry) and phase NG mode (internal symmetry) couple to one NG mode
- For skyrmion case in massless $\mathbb{CP}^1$ model, two translational NG modes, and phase NG mode and dilatation QNG mode couple to one NG mode(kelvin mode) and one NG-QNG mode
- Any couplings of NG modes satisfy Watanabe-Brauner relation